Mathematics and Statistics Vacation Scholarships Program

Geodesics on cones

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January, 2021

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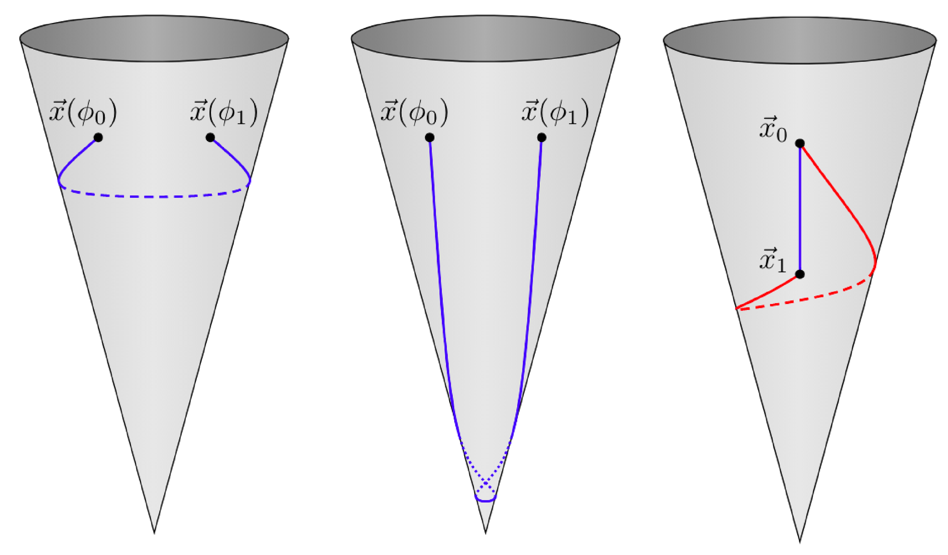
Introduction

Geodesics are sufficiently well-behaved curves $\vec{x}$ between two points on a manifold that extremise the length.
Variational calculus gives the equations governing geodesics, and hence we find geodesics on a number of surfaces.
We also investigate the winding and crossing of geodesics on the cone, and use its intrinsic lack of curvature to reduce the problem of geodesics on the cone to geodesics on the plane.

Permissible winding



Assuming $0 < \Delta\phi \leq \pi/2$, equation (3) makes sense exactly when

$$\left|\sin(\alpha)(\phi - \pi k)\right| < \frac{\pi}{2} \quad \forall \phi \in [-\Delta\phi, \Delta\phi + 2\pi k]$$
$$\iff \frac{-1}{2\sin(\alpha)} - \frac{\Delta\phi}{\pi} < k < \frac{1}{2\sin(\alpha)} - \frac{\Delta\phi}{\pi}. \quad (4)$$

The integers k satisfying (4) give the possible winding numbers for the geodesic between two points of the same height. For points of general position on the cone, mapping parametrised geodesics to the plane using T (see (5)) gives

$$\left|(\Delta\phi + 2\pi k)\sin(\alpha) + \Delta\phi\sin(\alpha)\right| < \pi$$

which is exactly equivalent to (4)! So permissible winding is not dependent on ρ_0 and ρ_1.

Variational calculus

A geodesic $\vec{x}$ must extremise the length functional

$$\mathcal{L}[x(t)] = \int_{t_0}^{t_1} \sqrt{g_{\mu\nu}(\vec{x}(t))\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}}\, dt, \quad (1)$$

where $ds^2 = g_{\mu\nu}(\vec{x})dx^\mu dx^\nu$ is the metric on the relevant manifold, and the integrand is the Lagrangian L. Variational calculus tells us that $\vec{x}$ is a stationary curve of $\mathcal{L}$ if and only if it satisfies the Euler-Lagrange equations, (2).

Curvature of the cone

Consider the bijection $T : \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}^+ \times \mathbb{R}$,

$$(\rho, \phi) \mapsto (\rho\csc(\alpha), \phi\sin(\alpha)) \quad (5)$$

mapping points on the cone to points on the plane. The infinitesimal line element is invariant under T, so the cone has no intrinsic curvature.

Now the Ricci curvature scalar R can be computed from parallel transport, which is given by the Levi-Civita connection derived from $g_{\mu\nu}$. Indeed, $R = 0$ at all points on the cone except the tip where it is undefined.

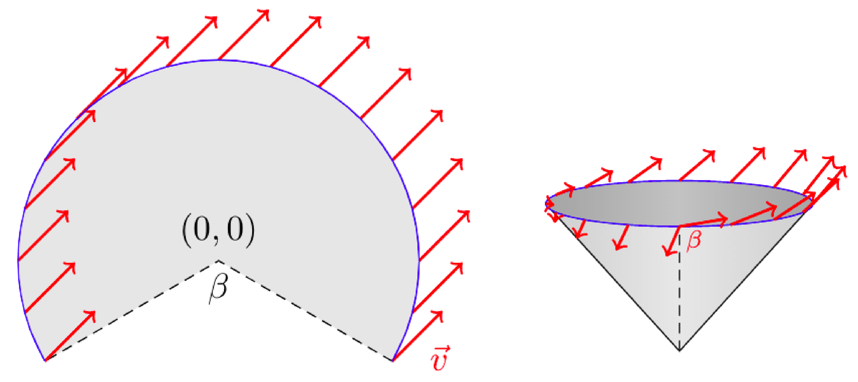


Figure 3: Parallel transport of a vector around the tip

Parallel transport of a vector around the vertex results in a rotation of $\beta = 2\pi(1 - \sin(\alpha))$. The *total curvature* of the cone can also be computed as β by considering the cone as the limit of a smooth surface. So all the curvature must be in the tip.

Computing geodesics

The Euler-Lagrange equations are given by

$$\frac{\partial L}{\partial \vec{x}} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\vec{x}}}\right] = \vec{0}. \quad (2)$$

These govern the form of geodesics. Note that we do reparametrise the geodesic $\vec{x}$ in (1) for some coordinates systems.

- In $\mathbb{R}^2$ with the Euclidean metric, geodesics take the form $\alpha x + \beta y = \gamma$. These are just straight lines, which matches intuition.
- On an infinite cylinder, geodesics from $\vec{x}(\phi_0) = (\phi_0, z_0)$ to $\vec{x}(\phi_1) = (\phi_1, z_1)$ look like
$$\frac{z - z_0}{z_1 - z_0} = \frac{\phi - \phi_0}{\phi_1 + 2\pi k - \phi_0}, \quad \phi \in [\phi_0, \phi_1 + 2\pi k]$$
for any $k \in \mathbb{Z}$, where k induces winding around the cylinder. These are helices.
- On the sphere, geodesics take the form
$$\phi(\theta) = \arccos(\alpha\cot(\theta)) + \beta.$$
These are arcs of great circles.
- On the cone, we first simplify (2) using symmetry. Geodesics between points $\vec{x}_0 = (\rho_0, -\Delta\phi)$ and $\vec{x}_1 = (\rho_0, \Delta\phi + 2\pi k)$ of the same height take the form
$$\rho(\phi) = \frac{\rho_0\cos(\sin(\alpha)(\Delta\phi + \pi k))}{\cos(\sin(\alpha)(\phi - \pi k))}, \quad (3)$$
where $\phi \in [-\Delta\phi, \Delta\phi + 2\pi k]$ for certain $k \in \mathbb{Z}$ (see (4) in Permissible winding). Here α is the angle the cone makes with the z-axis.

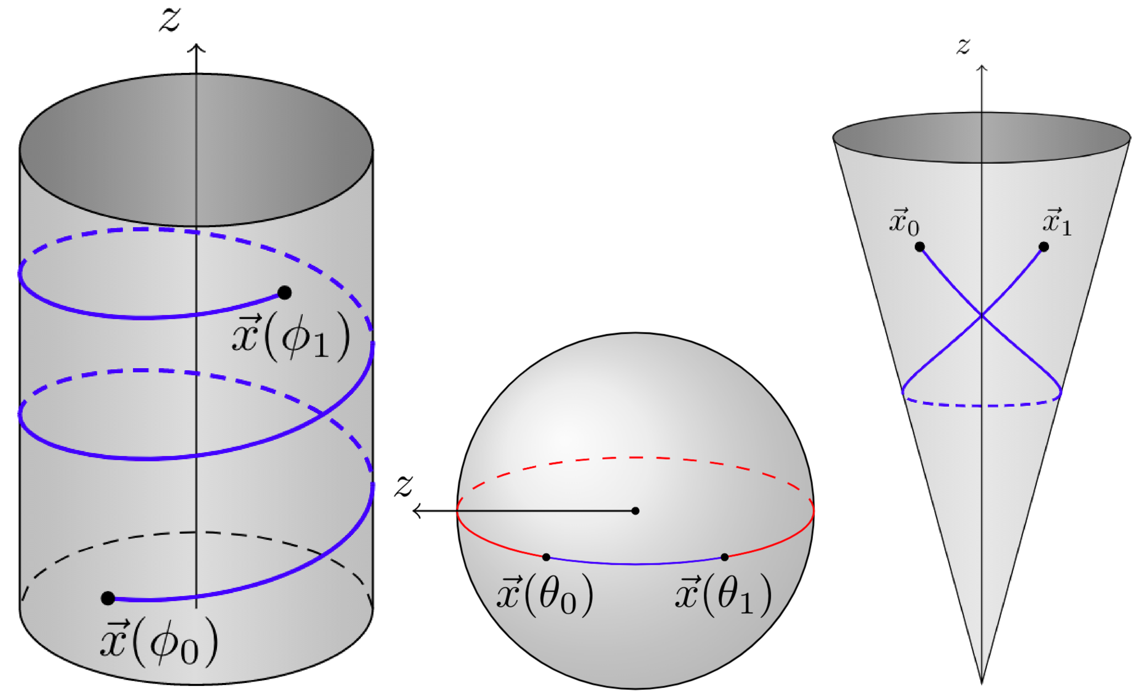


Figure 1: Geodesics on the cylinder, sphere and cone

Another approach

Since T locally preserves distance (see Curvature of the cone), parametrised geodesics on the cone are mapped to geodesics in the plane. So applying T^{-1} to the general straight-line in the plane, the geodesic between (ρ_0, ϕ_0) and (ρ_1, ϕ_1) is

$$\rho(\phi) = \frac{\rho_0\rho_1\sin((\phi_1 - \phi_0)\sin(\alpha))}{\rho_1\sin((\phi_1 - \phi)\sin(\alpha)) + \rho_0\sin((\phi - \phi_0)\sin(\alpha))},$$

where $\phi \in [\phi_0, \phi_1]$ includes desired winding. Viewing the geodesics in the plane also gives a nice way to understand the limitation (4) on winding, as we must avoid passing through the origin.

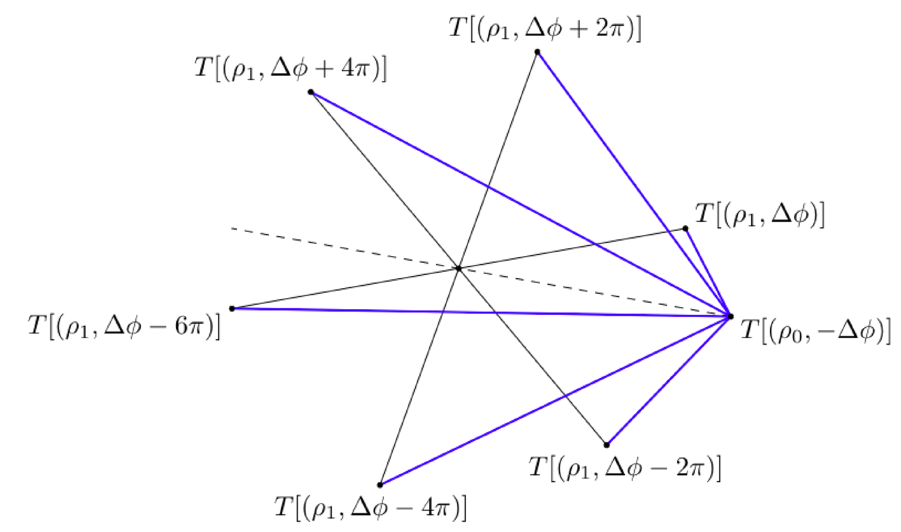


Figure 2: Geodesics mapped to the plane

Suggested readings

[1] P. Olver. Introduction to the calculus of variations. *University of Minnesota*, 2020.

[2] J. Atkins. Solving the geodesic equation. *University of Rochester*, 2018.

Counting crossings

$$C = \begin{cases} k & k \geq 0 \\ -k - 1 & k < 0 \end{cases}$$

Let C be the number of crossings a geodesic exhibits on the cone between points $(\rho_0, -\Delta\phi)$ and $(\rho_0, \Delta\phi + 2\pi k)$ of the same height. By considering (3), we get the formula shown to the left.

More information

This project was completed for the 2020-2021 Vacation Scholarship Program. I would like to thank Senior Lecturer Thomas Quella for his guidance, explanations, and corrections.

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