Partitions and Hilbert scheme of points on surfaces

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Introduction

The Hilbert scheme of n points on a surface is an interesting geometric object in topology and representation theory. The partitions of n are important combinatorical objects. There are certain links between these two concepts:

The set of $(\mathbb{C}^*)^2$-fixed points on Hilbert scheme of n points on $\mathbb{C}^2$, denoted by $Hilb_n(\mathbb{C}^2)$, is in one-to-one correspondence to the set of partitions of n.

We also study the homology group of $Hilb_n(\mathbb{C}^2)$. In particular, the Betti numbers of $Hilb_n(\mathbb{C}^2)$ are expressed in terms of partitions of n.

Partitions

A partition λ of a positive integer n is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$, and $\sum \lambda_i = n$. We use $\lambda \vdash n$ to represent that λ is a partition of n. The numbers λ_i are called the parts of the partition λ, whereas k is the number of parts of λ.

The generating series for $p(n) := \#\{\lambda \mid \lambda \vdash n\}$, i.e. total number of partitions of n, is given by:

$$P(q) = \sum_n p(n)q^n = \prod_{i=1}^{\infty} \frac{1}{1-q^i} = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + \cdots$$

Example 1. *For λ = 6 case, all the possible partitions are:*

$$(3,1,1,1), (2,2,2), (2,2,1,1), (1,1,1,1,1,1),$$
$$(4,1,1), (3,3), (3,2,1),$$
$$(6), (5,1), (4,2), (2,1,1,1,1).$$

They are represented by the following Young diagrams.

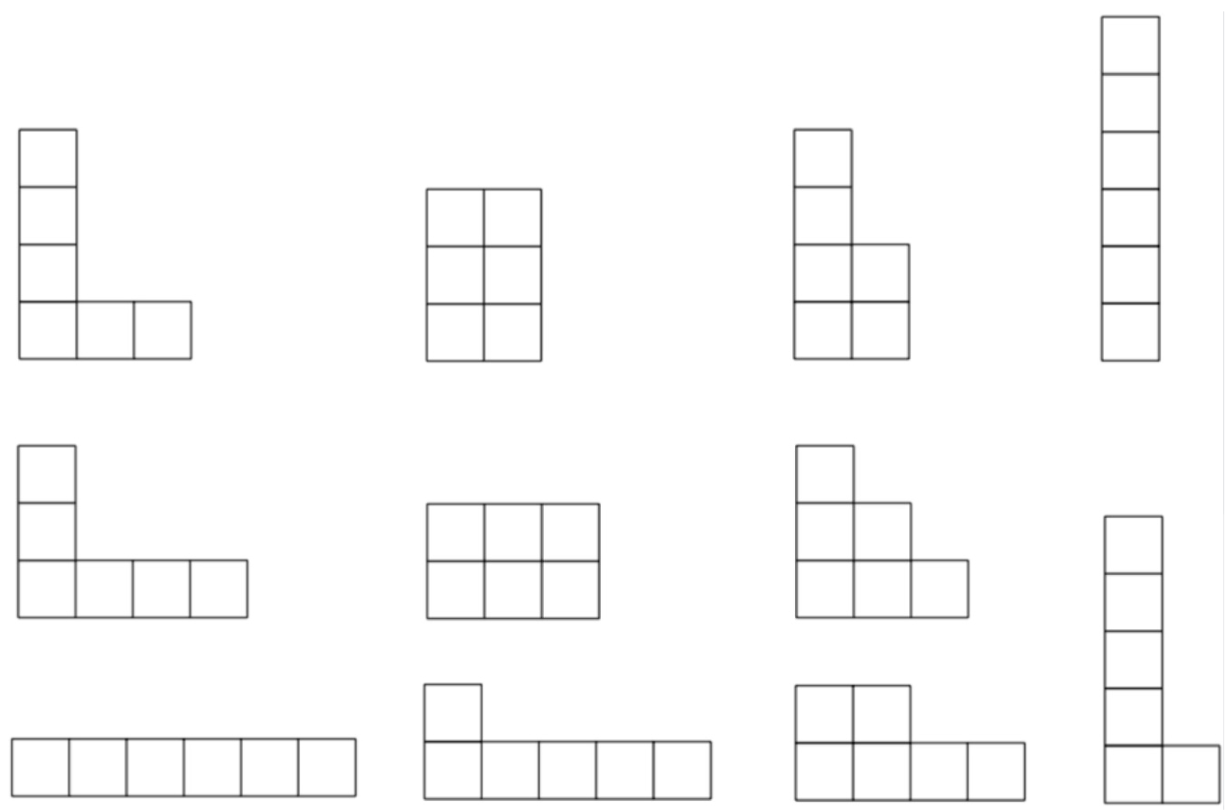


Figure 1: Young diagrams for $p(6)$ case

Therefore, the number $p(6) = 11$, matches with the coefficient of q^6 in the generating series of $P(q)$.

Hilbert scheme of points on $\mathbb{C}^2$

The Hilbert scheme $Hilb_n(\mathbb{C}^2)$ is defined to be

$$Hilb_n(\mathbb{C}^2) = \{\text{ideals } I \subset \mathbb{C}[x,y] \mid dim_{\mathbb{C}}(\mathbb{C}[x,y]/I) = n\}.$$

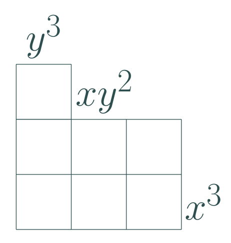
Recall an ideal I of $\mathbb{C}[x,y]$ is a subset of the polynomial ring $\mathbb{C}[x,y]$ such that

1. for any $g, h \in I$ then $g + h \in I$.
2. for any $f \in \mathbb{C}[x,y]$ and any $g \in I$ then $fg \in I$;

The *monomial ideals* are the ones generated by monomials with two variables x, y.

There exists a one to one correspondence between the monomial ideals of length n and the partitions of n. This is illustrated in the following example.

Example 2. *Consider the monomial ideal* $I := \langle x^3, xy^2, y^3 \rangle$*. One can associate it to the following partition*



Main results

In this section, we state some topological properties of the Hilbert scheme of points on $\mathbb{C}^2$. The space $Hilb_n(\mathbb{C}^2)$ is a resolution of singularity of the symmetric powers $(\mathbb{C}^2)^n/\mathfrak{S}_n$. In particular, $Hilb_n(\mathbb{C}^2)$ is smooth.

Let $H_k(Hilb_n(\mathbb{C}^2)) = H_k(Hilb_n(\mathbb{C}^2), \mathbb{Q})$ be the k-th homology group of $Hilb_n(\mathbb{C}^2)$ with rational coefficient. In particular, $H_k(Hilb_n(\mathbb{C}^2))$ is a $\mathbb{Q}$-vector space.

The Betti numbers b_k is defined to be

$$b_k := \dim_{\mathbb{Q}}(H_k(Hilb_n(\mathbb{C}^2))).$$

Theorem 1. *[N, Theorem A.8] The homology group*

$$H_*(Hilb_n(\mathbb{C}^2)) = \oplus_{k \in \mathbb{N}} H_k(Hilb_n(\mathbb{C}^2))$$

vanishes in odd degrees. The Betti number of $Hilb_n(\mathbb{C}^2)$ *is given by*

$$b_{2i} = \textit{number of partitions of } n \textit{ into } n-i \textit{ parts},$$
$$b_{2i+1} = 0.$$

Denote the Poincaré polynomial by

$$P_t(Hilb_n(\mathbb{C}^2)) = \Sigma_{i \geq 0} t^i b_i(Hilb_n(\mathbb{C}^2))$$

.

Theorem 2. *[N, Corollary A.9] The generating function of* $P_t(Hilb_n(\mathbb{C}^2))$ *is given by*

$$\sum_{n \geq 0} q^n P_t(Hilb_n(\mathbb{C}^2)) = \prod_{m=1}^{\infty} \frac{1}{1 - t^{2m-2}q^m}$$

Specializing at $t = 1$, we have

$$P_1(Hilb_n(\mathbb{C}^2)) = \sum_{i \geq 0} b_i(Hilb_n(\mathbb{C}^2)) = P(q).$$

Forthcoming Research

The previous story could be generalised to the $\mathbb{C}^3$ case, where we have the *plane partitions* and the *Hilbert scheme of points on* $\mathbb{C}^3$.

A plane partition is an array $\pi = (\pi_{i,j})_{i,j \geq 1}$ of non-negative integers such that π has finite supports (i.e. finitely many nonzero entries) and is weakly decreasing in rows and columns. If $\Sigma \pi_{i,j} = n$, then we write $|\pi| = n$ and say that π is a plane partition of n.

Similar to partitions on $\mathbb{C}^2$, plane partitions also have its generating series:

$$PP(q) = \sum_{n \geq 0} PL(n)q^n = \prod_{i=1}^{\infty} \frac{1}{(1-q^i)^i} = 1 + q + 3q^2 + 6q^3 + 13q^4 + 24q^5 + 48q^6 + \cdots,$$

where $PL(n)$ refers to the total number of plane partitions of n

Example 3. *The following is an example of the plane partition of 22,*

$$\begin{matrix} 5 & 4 & 2 & 1 & 1 \\ 3 & 2 \\ 2 & 2 \end{matrix}$$

It could be illustrated as follows:

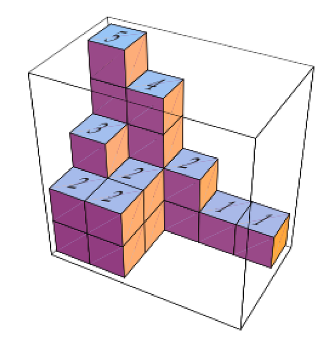


Figure 2

Here the numbers on the top boxes stand for the height in the z*-axis.*

Similar to $\mathbb{C}^2-$ case, Hilbert scheme of points on $\mathbb{C}^3$, $Hilb_n(\mathbb{C}^3)$, is define to be

$$Hilb_n(\mathbb{C}^3) = \{\text{ideals } I \subset \mathbb{C}[x_1,x_2,x_3] \mid dim_{\mathbb{C}}(\mathbb{C}[x_1,x_2,x_3]/I) = n\}$$

The monomial ideals in $Hilb_n(\mathbb{C}^3)$ are still parameterised by the plane partitions. However, there are several differences between $Hilb_n(\mathbb{C}^2)$ and $Hilb_n(\mathbb{C}^3)$. Unlike $Hilb_n(\mathbb{C}^2)$, $Hilb_n(\mathbb{C}^3)$ is not smooth, so we cannot take the usual homology group with the rational coefficient. There is a notion of homology group with coefficient in a "perverse sheaf" [RSYZ]. With the correct notion, we have similar results for $Hilb_n(\mathbb{C}^3)$.

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