Schwarzschild Solution and Black Hole Information Paradox

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Introduction

Black hole information paradox is one of the unsolved puzzles across different areas of physics and mathematics. This project is aiming to understand this paradox, and to approach this aim, we need go through several steps:

- Understand the mathematical tools needed to parameterize a spacetime.
- Understand the Schwarzschild solution for Einstein field equation, which describes the relationship between the spacetime and the stress-energy tensor.
- Understand Hawking radiation and the fact that black hole actually evaporates. This leads to the paradox, the information in the black hole seems to be lost, contradicting to ordinary quantum theory, where an all-seeing observer should be able to reconstruct everything.

Basic mathematics ingredients

A spacetime is modeled as a manifold, M, with a metric tensor, g. To understand this sentence, the following concepts are needed:

- Differentiable Manifold – describes the topology of the universe from the past to the future. A manifold is locally homeomorphic to $\mathbb{R}^n$.
  - Tangent space T_pM: a vector space for a given point p contains all tangent vectors passing through that point, and each tangent space is specific to its point, one cannot go directly from one tangent space to another.
  - Tensor T: a map $T : V^{*p} \times V^q \to \mathbb{R}$, where V is a vector space, and V^* is its dual vector space.
- Lorentzian manifold – is a special differentiable manifold that is locally homeomorphic to $\mathbb{R}^4$ with 1 time dimension and 3 space dimensions. It has an extra structure, metric tensor g, which has signature $(3, 1)$.
  - metric tensor g: a $(0, 2)$-tensor that $g : T_pM \times T_pM \to \mathbb{R}$, it describes how to calculate the inner product of the spacetime, hence the meaning of distance and angle.
  - connection/covariant derivative $\nabla_X T$: is the derivative of a vector field T in direction indicated by X. If $\nabla_X T = 0$, then the vector is parallel transported along the curve. It tells how we go from one tangent space to another. We define Christoffel symbols $\Gamma^c{}_{ab}e_c := \nabla_{e_b} e_a$, where e is a basis vector. It describes the changes of a basis vector with a given coordinate system. With the so-called Levi-Civita connection, we have

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad}(\partial_c g_{db} + \partial_b g_{dc} - \partial_d g_{bc}) \tag{1}$$

- Curvature
  - Riemann curvature $R^a{}_{bcd}$: is a $(1, 3)$-tensor that measures how much a vector deviates from its origin if we parallel transport it through a closed path, see figure 1. If the deviation is non-zero, the spacetime is curved, else it is flat.

$$R^a{}_{bcd} := R(dx^a, \partial_b, \partial_c, \partial_d) = \partial_c(\Gamma^a{}_{bd}) - \partial_d(\Gamma^a{}_{bc}) + \Gamma^a{}_{mc}\Gamma^m{}_{bd} - \Gamma^a{}_{md}\Gamma^m{}_{bc} \tag{2}$$

  - Ricci tensor $R_{bd} := R^m{}_{bmd}$: shows how size, like volume or area, of an object changes in some given direction indicated by the contraction of indices.
  - Scalar curvature $R := g^{ij}R_{ij}$: is the overall curvature without direction term.

All the details can be found in reference [1].

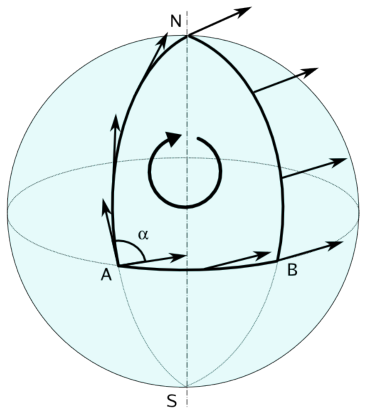

Figure 1: Parallel transport a vector at point A through a closed path on a sphere, the vector deviates. (From Stathis Kamperis, "How to derive the Riemann curvature tensor", https://ekamperi.github.io/mathematics/2019/10/29/riemann-curvature-tensor.html)

Schwarzschild solution for Einstein field equation

Consider the Einstein field equation

$$\underbrace{R_{\mu\nu}(g) - \frac{1}{2}R(g)g_{\mu\nu}}_{\text{spacetime geometry}} = \underbrace{\frac{8\pi G}{c^4}T_{\mu\nu}}_{\text{energy and mass}} \tag{3}$$

The right hand side is the stress-energy tensor, describing the distribution of mass and energy. All variables on the left hand side depend only on g. From equations in last section, both Ricci tensor $R_{\mu\nu}(g)$ and scalar curvature $R(g)$ are second derivative of g as Christoffel symbol is the first derivative of g. Therefore, Einstein field equation is a second-order PDE of g, which is extremely hard to solve.

For a spacetime containing a black hole, the mass is concentrated at the singularity of the black hole, so stress-energy tensor is 0 everywhere except the singularity. With two assumptions: 1) the black is static, independent to t; 2) the solution is spherical symmetric, we found the Schwarzschild solution g can be expressed in t and r, θ, ψ, the spherical coordinate system, with $r_s = \frac{2GM}{c^2}$, where G is the universal gravitational constant, M is the black hole's mass and c is the speed of light:

$$\begin{bmatrix} -(1-\frac{r_s}{r}) & 0 & 0 & 0 \\ 0 & (1-\frac{r_s}{r})^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2(\sin\theta)^2 \end{bmatrix} \tag{4}$$

r_s is the Schwarzschild radius, where the black hole's event horizon locates at. There are two singularities, but it turns out that only $r = 0$ is the true spacetime singularity, the manifold disappears at that point and scalar curvature $R(g)$ diverges. In contrast, $r = r_s$ is purely a coordinate singularity, which can be fixed by changing to a so-called Kruskal-Szekeres coordinates. This allows us to extend spacetime beyond $r = r_s$. One can pass the event horizon in one's own frame. The infinite spacetime can then be contracted into a finite graph called penrose diagram, figure 2. Each point in the penrose diagram corresponds to a 2-sphere, because the spacelike coordinate here is the radial coordinate and the angle coordinates are hidden. See the details in reference [2].

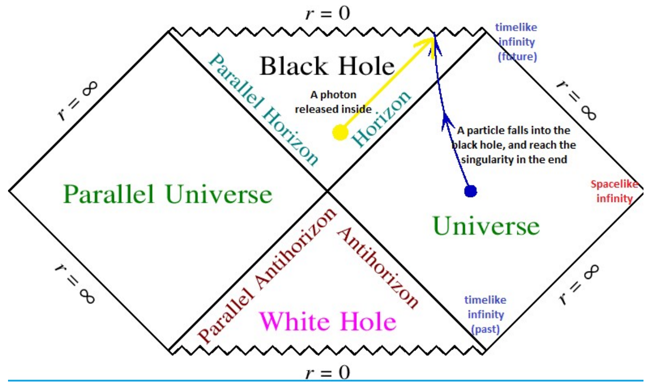

Figure 2: Penrose diagram for the extended Schwarzchild solution. (From Ro Jefferson, "Islands behind the horizon", https://rojefferson.blog/2021/07/05/islands-behind-the-horizon/)

See Figure 2, light travels at 45° in both diagram, if a photon is inside the event horizon, it can never escape. The white hole and the parallel universe, resulting from maximally extended Schwarzschild solution, are expected to be unphysical. A more realistic scenario describes a black hole formed by collapsing matter, see Figure 3.

Hawking radiation

We now consider a quantum field, which describes matter as elementary particles, on top of the Schwarzschild spacetime. By analogy to the Unruh effect it is found that for observers at spatial infinity, the black hole has a temperature

$$T = \frac{\kappa}{2\pi} \tag{5}$$

where $\kappa = \frac{1}{4M}$ is the surface gravity of the black hole. A black hole emits thermal radiations like a perfect black body, so energy can actually escape the black hole. This is called the Hawking radiation. Noting as the surface gravity is inverse of the mass, the black hole actually has negative heat capacity. Therefore, a Schwarzschild black hole cannot exist in a stable thermal equilibrium with an ordinary heat bath. The detailed mathematical derivation can be found in reference [4].

One significance is that this equation connects black hole mechanics with thermodynamics. Consider the following two equations

$$\frac{1}{8\pi}\kappa\Delta A = \Delta M - \Omega\Delta J, \text{ the first law of black hole mechanics} \tag{6}$$

$$T\Delta S = \Delta E + P\Delta V, \text{ the law of thermodynamics} \tag{7}$$

The entropy of the black hole, S_{bh}, is equal to $A/4$, where A is the surface area of the event horizon since we have $T = \frac{\kappa}{2\pi}$. Therefore, for a spacetime containing a static black hole, the total entropy is $S_{\text{total}} = S_{\text{bh}} + S_{\text{other}}$. More details about the validity of this total entropy can be found in reference [4].

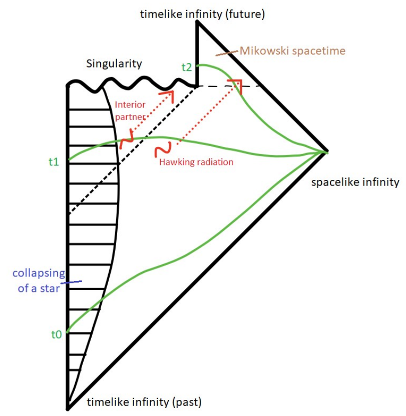

Figure 3: Penrose diagram for an evaporating black hole.

The information paradox

Consider an observer at spatial infinity, and slices of the spacetime at different instants of time t_0, t_1 and t_2 in figure 3. On each slices, there is a Hilbert space describing the quantum state, call them $\mathcal{H}_0$, $\mathcal{H}_1$ and $\mathcal{H}_2$.

At t_2, $\mathcal{H}_1$ decomposes into a tensor product of $\mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A$ describes the state inside the event horizon, and $\mathcal{H}_B$ describes the state outside. However, after the black hole evaporates completely at t_2, $\mathcal{H}_2$ can only depend on $\mathcal{H}_B$ as nothing can escape the event horizon and thermal radiation itself does not carry enough information about state inside. There cannot be a bijective linear map, $U : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_2$, describing time evolution such that nothing from the inside can affect anything outside. This is because the kernel of U is non-trivial, we have freedom on choosing state described by $\mathcal{H}_A$. This result contradicts the condition of unitary evolution in quantum theory, and leads to the paradox.

Reference [3] actually suggests two ideas on solving the paradox:

1. An extremely small core containing all the correlated information is left by the black hole after evaporation;
2. The information is hidden in correlations between particles emitted earlier and those emitted later.

References

References

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[2] Robert M. Wald. General Relativity.

[3] Robert M. Wald. Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics. The University of Chicago Press, 1994.

[4] Edward Witten. A mini-introduction to information theory. Rivista Nuovo Cimento (2020), 10 2019.

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