Dynamics of various inverted pendulum models

Eric Bai, supervised by Prof. Antoinette Tordesillas and Dr Edward Hinton

The University of Melbourne



Introduction

Simple, mechanical toys, like the inverted pendulum, have familiar and predictable dynamics. Complex systems, such as landslides (a geohazardous event where millions of rocks and debris slide down a slope[1]) have chaotic dynamical features. However, it is crucial to predict the time and location of future landslides, particularly in locations such as open pit mines. This project aims to understand the dynamics of the inverted pendulum, as dynamical similarities between it and actual landslide data could help model and predict future landslides.



Methodology

In this project the pendulum is either undamped or v^2 -damped, and either elastic or inelastic. Below is the $r - \theta$ coordinate system that is going to be used in the analysis. For the inelastic pendulums, r = l and for the elastic pendulum, r = l + x, where x is the extension of the spring, and l is the unstretched length. The equations will be nondimensionalised by setting $\hat{t} = \sqrt{\frac{q}{l}}t$ and $y = \frac{x}{l}$.



Inelastic, undamped pendulum

Newton's second law in the e_{θ} direction:

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = mg\sin(\theta$$

Rearranging and nondimensionalisation:

$$\theta'' = \sin(\theta)$$

Elastic, undamped pendulum

Newton's second law in both directions:

 $m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = mg\sin(\theta)$ $m(\ddot{r} - r\dot{\theta}^2) = -Kx - mg\cos(\theta)$

$$e_{r} = (\sin(\theta), \cos(\theta)), e_{\theta} = (\cos(\theta), -\sin(\theta))$$

$$\stackrel{\sim}{\underset{\sim}{r}} = re_{r}$$

$$\stackrel{\sim}{\underset{\sim}{r}} = \dot{r}e_{r} + r\dot{\theta}e_{\theta}$$

$$\stackrel{\sim}{\underset{\sim}{r}} = (\ddot{r} - r\dot{\theta})e_{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})e_{\theta}$$

Inelastic, damped pendulum

Newton's second law in the e_{θ} direction:

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = mg\sin(\theta) - D_0 \cdot sign(\dot{\theta})(l\dot{\theta})^2$$

Rearranging and nondimensionalisation:

(1)
$$\theta'' = \sin(\theta) - \delta sign(\theta')\theta'^2$$
(2)

Elastic, damped pendulum

Newton's second law in both directions:

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = mg\sin(\theta) - D_0\sqrt{\dot{r}^2 + r^2\dot{\theta}^2} r\dot{\theta}$$
$$a(\ddot{r} - r\dot{\theta}^2) = -Kr - mg\cos(\theta) - D_0\sqrt{\dot{r}^2 + r^2\dot{\theta}^2} \dot{r}$$

To investigate how these values of interest change as δ changes, plots were generated of these values vs δ .



Fitting analytical curves to this data allowed for analysis into what happens as $\delta \rightarrow \infty$:

$$\begin{array}{ll} 1.\theta_c \to \frac{\pi}{2} & 2.\theta'_c \to 0 \text{ like } \frac{1}{\sqrt{\delta}} \\ 3.\theta''_{max} \to 0 \text{ like } \frac{1}{2\delta} & 4.m_0 \to 0 \end{array}$$

This can be shown analytically. For large δ , we can assume that $\theta'' \approx 0$. Equation (1) can be re-written as

$$\theta' = \sqrt{\frac{\sin(\theta)}{\delta}}$$

This shows that for large δ :

maximum
$$\theta'$$
 occurs when $\sin(\theta) = 1$ $(\theta = \frac{\pi}{2})$
 $\theta'_c = \frac{1}{\sqrt{\delta}}$

Differentiating (9) gives that for large δ :

$$\theta'' = \frac{\sqrt{1 - \delta^2(\theta')^4}}{2\delta}$$

This shows that for large δ

•
$$m_0 = 0$$

• $\theta''_{max} = \frac{1}{2\delta}$.

Phase portrait of θ'' vs θ'



• Low β : $1 < \beta \le 1.755$

- Maximum velocity is reached when the spring is compressed
- There are two peaks for the velocity graph.
- As β increases, the second peak is becomes taller
- For $\beta < 1.6557$, the first peak is higher.
- Initial transitional period: $1.755 < \beta < 2.8$
- The phase portraits become shorter and wider
- The second peak dominates the first peak more and more
- The first peak becomes less noticeable
- Secondary transitional period: $2.8 \le \beta < 4$.
- The two peaks merge into one, and the phase portraits start to have kinks towards the end of the behaviour.
- This kink is gradually smoothed out.
- Large $\beta: \beta \ge 4$,

(9)

The kinks get smoothed out: curves approach the inelastic case.

Elastic, damped pendulum



$$m(r + r + r) = mx + mg\cos(r) + D_0 \sqrt{r} + r + r$$

Rearranging and nondimensionalisation:

Rearranging and nondimensionalisation:

$$\theta'' = \frac{1}{1+y}\sin(\theta) - \frac{2y'\theta'}{1+y}$$
(3)

$$\eta'' = \frac{1}{1+y}\sin(\theta) - \frac{2y'\theta'}{1+y} - \delta\sqrt{y'^2 + (1+y)^2\theta'^2} \theta'$$
(5)

$$y'' = (1+y){\theta'}^2 - \beta^2 y - \cos(\theta)$$
(4)

$$y'' = (1+y){\theta'}^2 - \beta^2 y - \cos(\theta) - \delta\sqrt{y'^2 + (1+y)^2\theta'^2} y'$$
(6)

Note in all cases $\beta = \sqrt{\frac{K/m}{g/l}}$ and $\delta = \frac{D_0 l}{m}$

This project will investigate the first-quadrant behaviour of the phase portraits of $\theta'' \vee s \theta'$ in response to close-to-stable initial conditions

$$\theta(0) = \epsilon = 0.001, \ \theta'(0) = 0, \ y(0) = \frac{-1}{\beta^2}, \ y'(0) = 0$$

The aim of this investigation is to obtain a detailed profile of how the dynamics changes as δ and β change, so that this model can be fitted to landslide data. Various points of interest are:

- θ'_c , which describes the maximum angular velocity, along with θ_c and t_c , describing the angular position and time at this instant.
- θ''_{max} , describing the maximum angular acceleration
- $frac_{max} = \frac{\theta' \operatorname{at} \theta''_{max}}{\theta' c}$, which describes the skew of the curves
- m_0 , describing the gradient at which the curve leaves the "origin"

Inelastic pendulum



Multiplying (1) by θ' and integrating with respect to \hat{t} , results in an energy equation (7). Applying initial conditions gives E = 1, which results in the analytical equation (8) and produces the values of interest below.

$$\frac{{\theta'}^2}{2} + \cos(\theta) = E$$
 (7) ${\theta''}^2 = {\theta'}^2 - \frac{{\theta'}^4}{4}$

(8)







- For large β , the inelastic damped case is recovered
- For $\delta = 0$, the elastic, undamped case is recovered
- Adding damping smoothes out the profile of the surface
- Damping does not affect the low β case as much as it affects the high β case
- $frac_{max}$ is controlled mainly by damping, until damping is large enough, and then it splits into two distinct β -regions.

References

- 1 What is a landslide and what causes one? (no date) U.S. Geological Survey. Available at: https://www.usgs.gov/faqs/what-landslide-and-what-causes-one (Accessed: 05 February 2024)
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