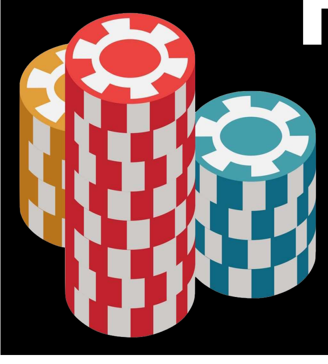

# Hitting probabilities in 3-player betting games


**Angel Yong He**, supervised by Prof. Mark Holmes

angelh1@student.unimelb.edu.au | **GitHub repo**: https://github.com/7angel4/3-person-betting-game

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## Introduction

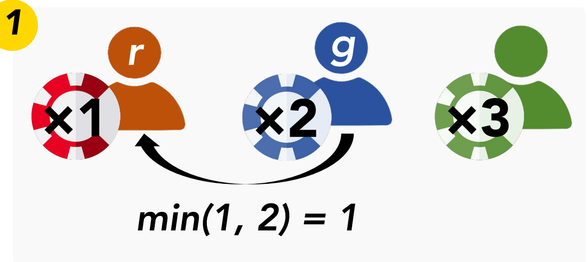
We study two variants of the models examined by Prof. Persi Diaconis [1]. Denote the players' fortunes at each round as $(X_n, Y_n, Z_n) \in \mathbb{N}^3$, which evolves as a Markov chain. Write $(X_0, Y_0, Z_0) = (x, y, z)$.

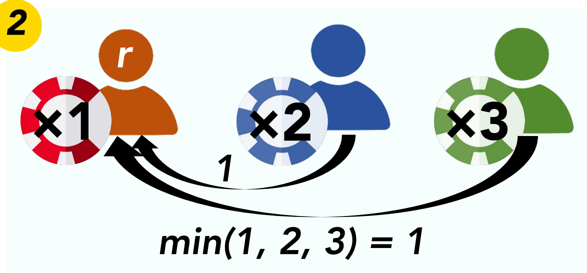
- **Game 1**: At each round, a giver and receiver are chosen at random. The giver transfers the minimum of their fortunes to the receiver.
- **Game 2**: Only a receiver is chosen each round to receive $\min(x, y, z)$.

Define the following terminology / representations:

- **Loser**: First player to reach 0 (if tied in Game 2, pick one randomly).
- **Winner**: First player to have all the money.
- $L_{(x,y,z)}$: Probability that player 1 loses, given initial state $(x, y, z)$.

The research problem addressed is the ***difficulty of attaining $L_{(x,y,z)}$ by hand***. This poster presents an overview of the program I produced in response, and some further analysis on the first-hand data.




## Problem Specification

While $P(\text{winner} = \text{player } 1)$ is simply $\frac{x}{x+y+z}$ [2], determining their losing probability is much more difficult. *E.g. Consider the initial state* ***(1, 2, 3)***:

**1**

$$\begin{aligned} L_{(1,2,3)} &= \frac{1}{6}\big(L_{(0,3,3)} + L_{(0,2,4)} + L_{(2,1,3)} \\ &\quad + L_{(1,0,5)} + L_{(2,2,2)} + L_{(1,4,1)}\big) \\ &= \frac{1}{6}\left(1 + 1 + L_{(2,1,3)} + 0 + \frac{1}{3} + L_{(1,4,1)}\right) \\ &= \frac{7}{18} + \frac{1}{6}\big(L_{(2,1,3)} + L_{(1,4,1)}\big) \end{aligned}$$

One step of the analysis has already introduced 6 intermediate states (but some can be 'pruned' early).

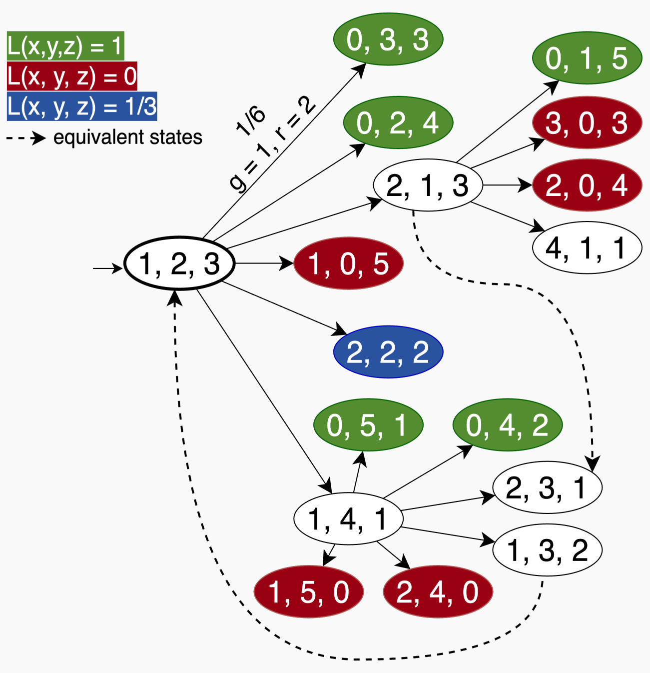



*Figure 1.1: Modified transition diagram for Game 1, illustrating how the program analyses state (1, 2, 3).*

**2**

$$\begin{aligned} L_{(1,2,3)} &= \frac{1}{3}\big(L_{(3,1,2)} + L_{(0,4,2)} + L_{(0,1,5)}\big) \\ &= \frac{1}{3}\big(L_{(3,1,2)} + 1 + 1\big) \\ &= \frac{1}{3}L_{(3,1,2)} + \frac{2}{3} \end{aligned}$$

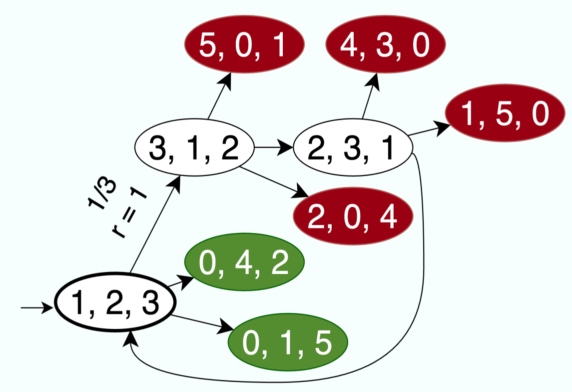



*Figure 1.2: Modified transition diagram for Game 2.*

Due to the 'depth' of the analysis, as the states become larger, it quickly becomes intractable to calculate by hand. Thus, I wrote a Python program to automate the analysis (with guaranteed termination) and produce the exact $L_{(x,y,z)}$'s in fraction form.

## Program Functionalities

Given the initial state as input, the program can:

- Generate and solve the first step analysis equations;
- Generate $L_{(x,y,z)}$ for the initial and intermediate states.
- Export the equations and probabilities to a text file.

These are combined into the `LoserAnalysis` class.
Analysis in the "Selected Results" section is also automated.

## Implementation Details

- The program uses the **`sympy`** library to solve equations.
- $L_{(x,y,z)} = L_{(x,z,y)}$. Our convention is to store the smaller stack at front.
- To accommodate for slow execution large inputs, the program offers:
  - i. **Fallback option**: Approximates $L_{(x,y,z)}$, by *enumerating* the game up to a fixed number of rounds ($t$), using **memoisation** to optimise efficiency. The result is accurate to $2^{-t}$.
  - ii. **Exception-handling**: If the maximum recursion depth or time limit is hit, the program throws and catches an exception, then skips the current state / uses the fallback method.

## Selected Results

The probabilities generated are stored as CSV files in the project's GitHub repository. We only present some remarkable results here.

### 1. If player 1's objective is not to lose, when are they better off giving away \$1?

**1** The smallest* such state is **(11, 19, 48)** vs. **(10, 19, 48)**:

$L_{(11,19,48)} =$ 9505533830767552681325332460536243296492851849399342033764290735641529081784363354971001546676377024313257419331316917
1878567165664636090433413549346556542318405224485002394984381187095762857031875372514851648744689876606583868833130789124944281795228465
4738673214410277893782426223384701540290921143078273287839698880724012451250170055782308083321973435371340847387859216007087622235865621
6355262548657371387855499893220300767863500018989926419644556351560167468150557905546740168393234581650793331708269728862569517012615864
81180710168736322560935541579312606018598919769976414734737645143600881893716186589372178139625701941/1781977679341565653879062316696873
8432477569983271536712319868523149518433213441869248413988212864124407801562834940404112578706727560878477149731918101076780906408169254
2704158384981828417284705119427008399746104598790307294201748170143987149291135520733374002287217775101669536582256671886738481160720021
5463380494510770187945490845928143445232732949053965144678241474218200604350499913285920102311013763668735307552502191738215421222671791
49363687922703404603633363600205029257226017910414902297593134822746299465920848726683501707045280321742201920572234233639669931523977995
36216014235688392527706247314395749592323268734337 $\approx 0.5334260883828699$

$L_{(10,19,48)} =$ 384927437348092865826787154522036966929331477671257388795406213322077025027329708904002230160514488758925861008529937
2641865076307695712428629862510743966379144258747669183765648308949640633266498014775160083551908074040886977133195582955400146709753989
3736472089107670746944940138298692043311547964288456867952310747031559711923401123354938125594417587908704419279778529647788014939856406
6044668357513633952588377925634818352400474258882404223657377314751413083100101954283316910145455750718339182097013355811579429030093387
81392941800/7217263194771645633240126555255838476299049886246841503769225103997039449052747771980511648002392515987375937925303682666479
7405000819352675795582182050749468591950783638253168541197856184516316883369948927113568339288396113210508323149280922041533365800656183
4637915106023376987127445506182008039880270393859346595996143509931738007423070998446360400532405859776577658805815263294367324760378016
5125261773026254548974238037597139908155084767333005841357760140215127815456879580085371803405945673770446916448716439745218669565734873
6787 $\approx 0.5333426632230114$

**2** The smallest* such state is **(4, 5, 11)** vs. **(3, 5, 11)**:

$$L_{(4,5,11)} = \frac{164}{243} \approx 0.6748971193415638$$

$$L_{(3,5,11)} = \frac{2}{3} \approx 0.6666666666666667$$

* by lexicographical ordering

### 2. Similarly, when is player 1 better off giving away \$1 to another player?

The smallest such state is **(2, 4, 5)** vs. **(1, 5, 5)**:

$$L_{(2,4,5)} = \frac{3529997}{7036351} \approx 0.5016800611566989$$

$$L_{(1,5,5)} = \frac{3522116}{7036351} \approx 0.5005600203855664$$

The smallest such state is **(2, 9, 10)** vs. **(1, 10, 10)**:

$$L_{(2,9,10)} = \frac{327195}{368089} \approx 0.888901868841503$$

$$L_{(1,10,10)} = \frac{8}{9} \approx 0.8888888888888889$$

### 3. Visualisation of $L_{(x,y,z)}$ for varying $x$, fixed $y$ and $z$

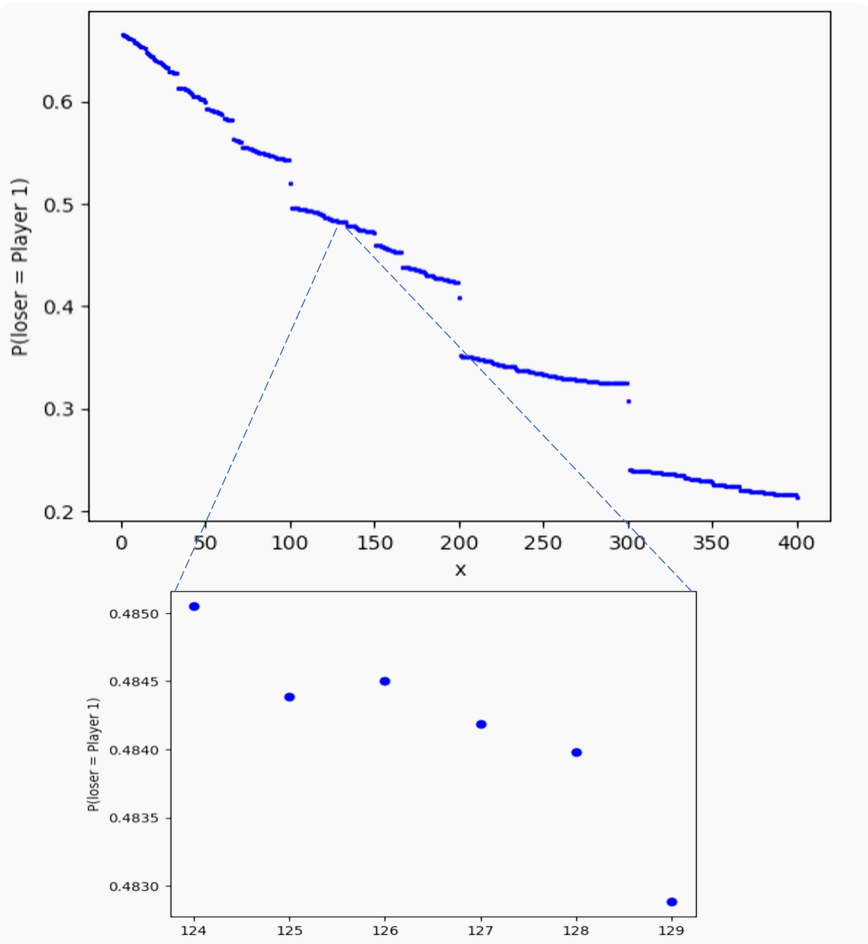



*Figure 2.1: $L_{(x,y,z)}$ for $x \in [1, 400]$, $y = 200$, $z = 300$, zoomed into the range $x \in [124, 129]$.*

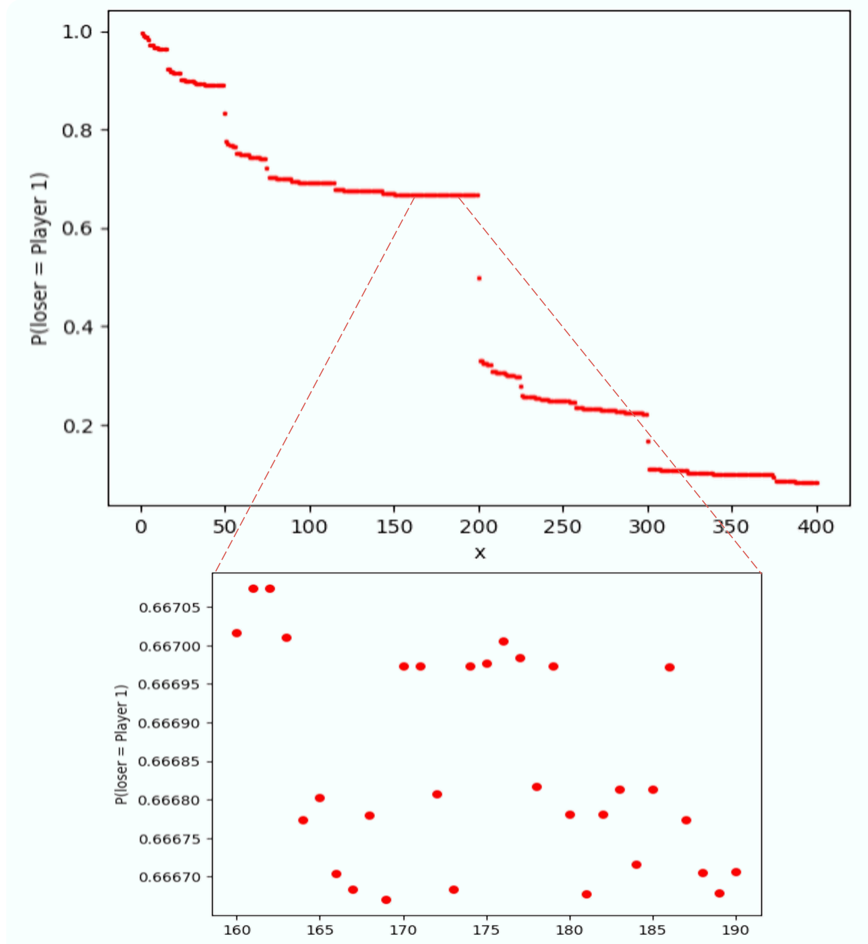



*Figure 2.2: $L_{(x,y,z)}$ for $x \in [1, 400]$, $y = 200$, $z = 300$, zoomed into the range $x \in [160, 190]$.*

### 4. Visualisation of $L_{(x,y,z)}$ for fixed sum, $x + y + z$

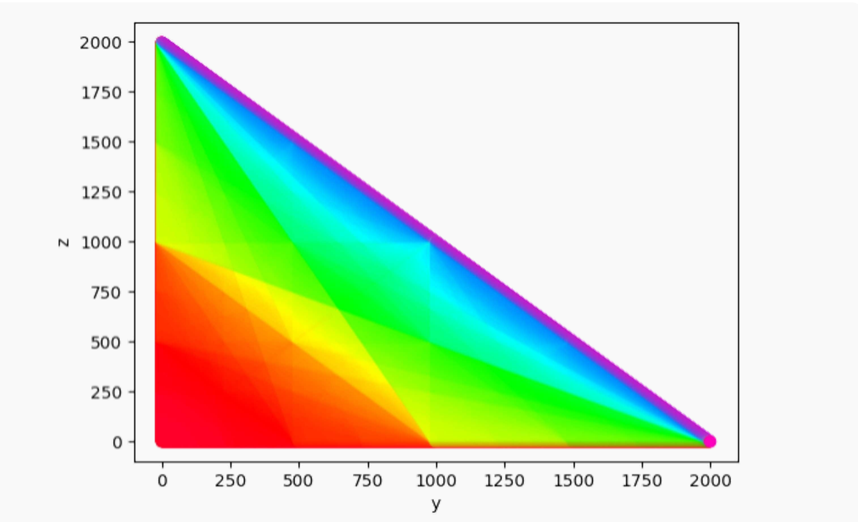



*Figure 3.1: $L_{(x,y,z)}$ for fixed $x + y + z = 2000$ in Game 1.*

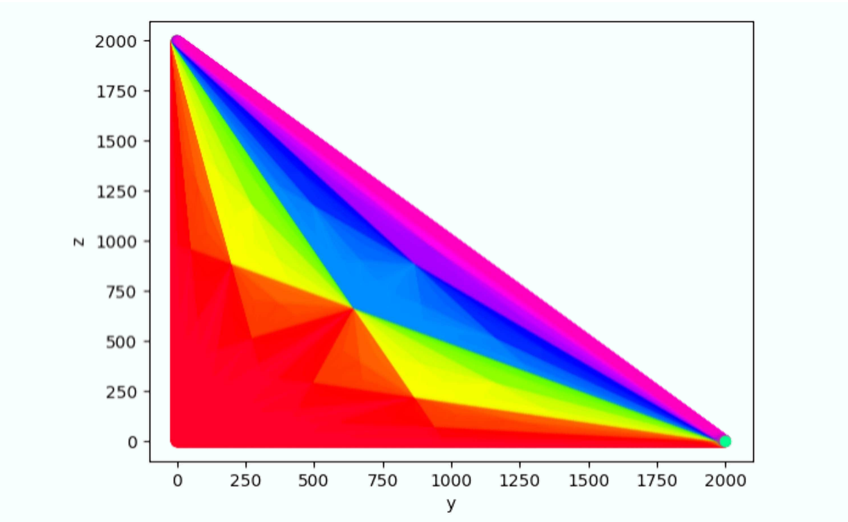



*Figure 3.2: $L_{(x,y,z)}$ for fixed $x + y + z = 2000$ in Game 2.*

## Key Observations

- The big **jumps** happen when player 1's fortune is tied with another player's fortune.
- The function $L_{(x,y,z)}$ is **not monotonic decreasing in $x$** (visible when we zoom into a flat segment).
- Given the same $(x, y, z)$, the program executes **faster on Game 2** than Game 1 (no. of intermediate states grows exponentially with base 6 in Game 1, but with base 3 in Game 2).
- **Game 2** has **more states where player 1 is not better off with an extra \$1** (more flat segments).
- For both models, $L_{(x,y,z)}$ is **fractal-like**!

## Future extensions

- Variants of the model:
  - i. Players choose at random to give $\min(x, y)$ or $\min(x, y, z)$.
  - ii. More than 3 players.
- Optimise program's efficiency for large inputs.

## Acknowledgement


This project has really taught me how to effectively integrate theory with practice, computing techniques with mathematical analysis, to conduct research. I am glad that my computing skills can be incorporated to produce some concrete results. Many thanks to my supervisor, **Prof. Mark Holmes**, for his insightful guidance and support throughout this project.


## References


[1] Diaconis, P. & Ethier, S. (2020). Gambler's Ruin and the ICM. *Statistical Science, Statist. Sci. 37*(3), 289-305. https://doi.org/10.1214/21-STS826

[2] Grinstead, C. M. & Snell, J. L. (2006). 12.2: Gambler's Ruin. In Doyle, P. G (Ed.), *Introductory Probability* (pp. 487-490). American Mathematical Society.