# Enumerating Alternating Sign Matrices via Six-vertex Models

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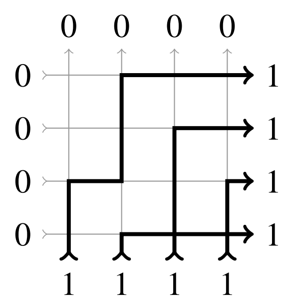
## Introduction

An interesting application of the six-vertex model is found in the enumeration of Alternating Sign Matrices (ASMs). ASMs are defined as square matrices where each element can be a 0 a 1 or a $-1$, each row and column must sum to 1, and the nonzero entries alternate in sign —hence the name.

It turns out that there is a bijection between the $m\times m$ ASMs and $m\times m$ lattices in the six-vertex model, and similarly between the $m\times m$ triangular lattices and $m\times m$ diagonally symmetric ASMs. Here is an example of a $4\times 4$ ASM, and the corresponding six-vertex lattice:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \longleftrightarrow$$




This bijection is useful as the enumeration of ASMs and their symmetry classes by standard means is a very difficult problem. In this section we will explore how partition functions of six-vertex models can be used to find this enumeration.

## The Ice Point

The first step in achieving this enumeration from our partition functions is finding a choice of variables such that each of the Boltzmann weights are equal to 1. This evaluation point is referred to as the 'ice point'.

Here, each lattice has a contribution of 1 to the partition function, making it simply count the number of valid lattices. By the bijection, this is also an enumeration of the ASMs. For the square lattice, this ice point is:

$$x_i = 1, \quad y_i = q, \quad q = e^{\frac{2\pi}{3}i}$$

Whereas the triangular lattice has ice point:

$$x_i = e^{\frac{\pi}{3}i}, \quad q = e^{\frac{2\pi}{3}i}, \quad p = -e^{\frac{2\pi}{3}i}, \quad r = -1$$

These values make all of the vertex weights one; however they result in the following relation for the corner weights:

$$t_1\left(e^{\frac{\pi}{3}i}\right)^2 = t_2\left(e^{\frac{\pi}{3}i}\right)\times t_3\left(e^{\frac{\pi}{3}i}\right) = t_4\left(e^{\frac{\pi}{3}i}\right)^2 = \frac{1}{3}$$

We can, however, remove this effect with a renormalisation, as valid lattices will have an equal number of $t_2$ corners and $t_3$ corners.

This is because these lattices as a whole are path preserving, where each node has the same number of 1s going in and out. However the $t_2$ corner creates a path, as it has a 0 going in and a 1 going out, while $t_3$ corners destroy a path. Since everything else in the lattice is path preserving, we must have an equal number of corners creating paths as we have corners destroying paths. So if we renormalise $T_m$ by $3^{\frac{m}{2}}$ we will get our desired enumeration.

## Schur Polynomials

One way to evaluate our functions is by first expressing them in terms of Schur polynomials, which form a basis for the set of symmetric polynomials in $n$ variables.

Because our partition functions are symmetric, we expect them to have an expression in these Schur polynomial. Given an integer partition $\lambda$, the corresponding Schur polynomial is defined as:

$$S(\lambda; x_1 \ldots, x_n) = \frac{\det\begin{bmatrix} x_1^{\lambda_1+n-1} & \cdots & x_n^{\lambda_1+n-1} \\ x_1^{\lambda_2+n-2} & \cdots & x_n^{\lambda_2+n-2} \\ \vdots & \ddots & \vdots \\ x_1^{\lambda_n} & \cdots & x_n^{\lambda_n} \end{bmatrix}}{\prod_{1\le i<j\le n}(x_i - x_j)}$$

$\lambda$ is an ordered set of $n$ weakly decreasing positive integers.

To represent our determinant formula as a Schur polynomial we will first perform the rescaling $y_i \to q x_{i+m}$. The motivation behind this rescaling is that Schur polynomials have a known evaluation when all variables are sent to 1, which we can exploit at the ice point. From inspection, it appears that our partition function $Z_m$ becomes:

$$Z_m(x_1,\ldots,x_{2m}) = \frac{(1-q)^m q^{\frac{m}{2}} \prod_{i=1}^{2m} x_i^{\frac{1}{2}}}{(-q)^{\binom{m}{2}} \prod_{i,j=1}^{m} x_i - q^2 x_{j+m}} S(\lambda; x_1,\ldots,x_{2m})$$

with $\lambda = \{m-1, m-1, m-2, m-2, \ldots, 1, 1, 0, 0\}$. We must now show that this formula satisfies all prior recursion relations, and hence is equal to the original partition function.

The symmetry, degree and vanishing of this function at $x_i \to 0$ is consistent with the original partition function. Hence, we turn our attention to specialisation 2.

To show the $Z_m(x_1,\ldots,x_{2m})|_{x_i=y_j}$ specialisation we first recognise that it is sufficient to show $Z_m(x_1,\ldots,x_{2m})|_{x_1=y_m} = Z_{m-1}(x_2,\ldots,x_{2m-1})$ as we take the symmetry of the function as given. We examine what happens to the determinant in the Schur polynomial under this specialisation. Noting that since $q = e^{\frac{2\pi}{3}i}$ we have that $q^3 = 1$ and thus we perform column operations to rewrite the determinant as follows:

$$\begin{bmatrix} (qx_{2m})^{3m-2} & x_2^{3m-2} & \cdots & x_{2m-1}^{3m-2} & x_{2m}^{3m-2} \\ (qx_{2m})^{3m-3} & x_2^{3m-3} & \cdots & x_{2m-1}^{3m-3} & x_{2m}^{3m-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (qx_{2m})^4 & x_2^4 & \cdots & x_{2m-1}^4 & x_{2m}^4 \\ (qx_{2m})^3 & x_2^3 & \cdots & x_{2m-1}^3 & x_{2m}^3 \\ qx_{2m} & x_2 & \cdots & x_{2m-1} & x_{2m} \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix} \to \begin{bmatrix} x_{2m}^{3m-2} & x_2^{3m-2} & \cdots & x_{2m-1}^{3m-2} & 0 \\ 0 & x_2^{3m-3} & \cdots & x_{2m-1}^{3m-3} & x_{2m}^{3m-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{2m}^4 & x_2^4 & \cdots & x_{2m-1}^4 & 0 \\ 0 & x_2^3 & \cdots & x_{2m-1}^3 & x_{2m}^3 \\ x_{2m} & x_2 & \cdots & x_{2m-1} & 0 \\ 0 & 1 & \cdots & 1 & 1 \end{bmatrix}$$

We can now make the following observations about our expression.

- It is a polynomial in $x_{2m}$ of degree $6m-5$.
- It is 0 if $x_{2m}$ is equal to any of 0, $x_i$, $e^{\frac{2\pi}{3}i}x_i$ or $e^{\frac{4\pi}{3}i}x_i$ for $i \neq 1, 2m$.
- We have thus found $6m-5$ roots and evaluated our expression up to a term $C(x_2,\ldots,x_{2m-1})$ independent of $x_{2m}$.

We can thus write:

$$C(x_2,\ldots,x_{2m-1}) = \frac{\det\begin{bmatrix} x_{2m}^{3m-2} & x_2^{3m-2} & \cdots & x_{2m-1}^{3m-2} & 0 \\ 0 & x_2^{3m-3} & \cdots & x_{2m-1}^{3m-3} & x_{2m}^{3m-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{2m}^4 & x_2^4 & \cdots & x_{2m-1}^4 & 0 \\ 0 & x_2^3 & \cdots & x_{2m-1}^3 & x_{2m}^3 \\ x_{2m} & x_2 & \cdots & x_{2m-1} & 0 \\ 0 & 1 & \cdots & 1 & 1 \end{bmatrix}}{x_{2m}\prod_{i=2}^{2m-1}(x_i - x_{2m})(e^{\frac{2\pi}{3}i}x_i - x_{2m})(e^{\frac{4\pi}{3}i}x_i - x_{2m})}$$

Since $C(x_2,\ldots,x_{2m-1})$ does not depend on $x_{2m}$ we are free to take $x_{2m} \to 0$. We now perform the following simplifications

- We cofactor expand down the left most column.
- All cofactors vanish apart from the factor of $x_{2m}$ which cancels.
- We cofactor expand down the right most column.
- All cofactors vanish apart from the factor of 1.
- We bring a factor of $x_i^3$ out of each remaining column which cancels with our denominator.

Leaving us with:

$$C(x_2,\ldots,x_{2m-1}) = \det\begin{bmatrix} x_2^{3m-5} & \cdots & x_{2m-1}^{3m-5} \\ x_2^{3m-6} & \cdots & x_{2m-1}^{3m-6} \\ \vdots & \ddots & \vdots \\ x_2^4 & \cdots & x_{2m-1}^4 \\ x_2^3 & \cdots & x_{2m-1}^3 \\ x_2 & \cdots & x_{2m-1} \\ 1 & \cdots & 1 \end{bmatrix}$$

which is the same determinant as we begin with but with $m \to m-1$ and missing $x_1$ and $x_{2m}$. Plugging this into our expression and performing some algebra gives us the exact specialisation we want, which may be quickly verified .
Now that we know our Schur polynomial formula is valid, we can evaluate it at $(x_1,\ldots,x_{2m}) = (1,\ldots,1)$. We use the following known evaluation for Schur polynomials:

$$S(\lambda; 1,\ldots,1) = \prod_{1\le i<j\le n}\frac{\lambda_i - \lambda_j + j - i}{j-i}.$$

Using this as well as our value for $q$ and then simplifying leaves us with the following evaluation for the number of $m\times m$ ASMs:

$$Z_m(1,\ldots,1) = \left(\frac{3}{4}\right)^{\binom{m}{2}} \prod_{i,j=1}^{m}\frac{3(j-i)+1}{2(j-i)+1}$$

However when we move our attention to the triangular lattice, $T_m$, we find that this method is not nearly as effective. Unlike the square lattice, the partition function for the triangular lattice is not a single Schur polynomial and is instead a complicated linear combination of Schur polynomials. Determining this is beyond the scope of this project.

## Factor Exhaustion

Another way we can evaluate our partition function is using the method of factor exhaustion, which involves determining something known to be a polynomial by finding all of its factors. We define the Matrix

$$M(x,y) = \left(\frac{y^{\frac{1}{2}(m+j-i)} - y^{-\frac{1}{2}(m+j-i)}}{x^{\frac{1}{2}(m+j-i)} - x^{-\frac{1}{2}(m+j-i)}}\right)_{1\le i,j\le m}$$

We observe that the expansion of $\det M(x,y)$ as a function of $y$ consists of a sum of monomials with either all irreducible half powers, if $m$ is odd, or all whole powers if $n$ is even. Thus the following expression will be a polynomial in the variable $y$ of degree $m^2$:

$$y^{\frac{m^2}{2}} \det M(x,y)$$

We now observe what happens when we send $y \to x^k$ with $0 \le k < m$. We notice the identity via telescoping :

$$\frac{x^{\frac{k}{2}(m+j-i)} - x^{-\frac{k}{2}(m+j-i)}}{x^{\frac{1}{2}(m+j-i)} - x^{-\frac{1}{2}(m+j-i)}} = \sum_{\substack{l=1-k \\ \text{steps of 2}}}^{k-1} x^{\frac{l}{2}(m+j-i)}$$

Using this identity we can decompose our matrix into a sum of matrices:

$$M(x^k,x) = \sum_{\substack{l=1-k \\ \text{steps of 2}}}^{k-1} \left(x^{\frac{l}{2}(m+j-i)}\right)_{1\le i,j\le m}$$

We see that each of these $k$ sub matrices have rank 1. It follows that the rank of $M(x^k,x) \le k$, thus $\det M(x^k,x) = 0$ for all $0 \le k < m$.

We can then conclude that our previously mentioned polynomial has roots at $y = x^k$ of multiplicity at least $m-k$. A similar argument can be made for $y = x^{-k}$. This gives us the $m^2$ roots, which means we have now defined the polynomial up to a factor $C(x)$, independent of $y$. To determine this we note the following equation:

$$C(x) = \frac{y^{\frac{m^2}{2}} \det M(x,y)}{\prod_{i,j=1}^{m}(x^{i-j} - y)}$$

Since $C(x)$ is not dependent on $y$ we consider what happens as $y \to \infty$. With some limit arguments and some rearranging we can show that:

$$C(x) = (-1)^{m^2} x^{-\frac{m(m+1)}{2}} \det\left[\frac{1}{x^{\frac{1}{2}(m-2i)} - x^{-\frac{1}{2}(m+2j)}}\right]_{1\le i,j\le m}$$

This can be evaluated as a special case of the following identity where $x_i = x^{\frac{1}{2}(m-2i)}$ and $y_j = x^{-\frac{1}{2}(m+2j)}$:

$$\det\left[\frac{1}{x_i - y_j}\right]_{1\le i,j\le m} = \frac{\prod_{1\le i<j\le m}(x_i - x_j)(y_j - y_i)}{\prod_{i,j=1}^{m} x_i - y_j}$$

Which can be proven using factor exhaustion and is omitted for brevity. Putting this all together we get the following identity:

$$\det M(x,y) = (-1)^{\frac{m}{2}(5m+1)} \frac{\prod_{i,j=1}^{m}(x^{i-j} - y)\prod_{\substack{i,j=1 \\ i\neq j}}^{m}(x^i - x^j)}{y^{\frac{m^2}{2}} x^{\frac{m}{2}(2m^2+m-1)} \prod_{i,j=1}^{n} x^{\frac{1}{2}(m-2i)} - x^{-\frac{1}{2}(m+2j)}}$$

We now wish to apply this identity to the partition function $Z_m$. We begin by considering the specialisations $x_i \to x^i$ and $y_i \to qx^{m+i}$. Recalling that $q = e^{\frac{2\pi}{3}i}$ ,the determinant of our partition function becomes:

$$\left[\frac{1}{(x^i - qx^{j+m})(x^i - q^2x^{j+m})}\right]_{1\le i,j\le m} \to \left[\frac{1}{x^{2i} + x^{j+m+i} + x^{2(j+m)}}\right]_{1\le i,j\le m}$$

We now observe the following relation by telescoping:

$$x^{2i} + x^{j+m+i} + x^{2(j+m)} = \frac{x^{3(j+m)} - x^{3i}}{x^{j+m} - x^i} = x^{j+m+i} \times \frac{x^{\frac{3(j+m-i)}{2}} - x^{-\frac{3(j+m-i)}{2}}}{x^{\frac{j+m-i}{2}} - x^{-\frac{j+m-i}{2}}}$$

Plugging this in, pulling a factor of $x^{-i}$ out of each row and $x^{-j-m}$ out of each column and then simplifying gives us:

$$\prod_{i=1}^{m} x^{-2i-m} \det\left[\frac{x^{\frac{j+m-i}{2}} - x^{-\frac{j+m-i}{2}}}{x^{\frac{3(j+m-i)}{2}} - x^{-\frac{3(j+m-i)}{2}}}\right]_{1\le i,j\le m}$$

We recognise that the inside of our determinant is equivalent to $M(x^3,x)$. We can thus apply our identity from earlier and plug this into $Z_m$. We then use L'Hopital's rule to take $x \to 1$ leaving us with the following expression for the number of $m\times m$ ASMs after some simplification:

$$Z_m(1,\ldots,1) = (-3)^{\binom{m}{2}} \prod_{i,j=1}^{m}\frac{3(j-i)+1}{m+j-i}$$

Which is equivalent to our earlier expression.
Sadly applying this method to $T_m$ is much harder as the function inside the Pfaffian is significantly more complicated then the function inside the determinant of $Z_m$.

## Conclusion

In the pursuit of evaluating $Z_m$ we have learnt about two techniques that see wide use in the study of integrable models. We have seen how useful they can be in transforming an equation which takes a complicated form into one far more simple and easily computable at a point of interest. However we have also encountered the greater problem of evaluating $T_m$, which remains an open problem to this day as it seems impervious to the methods currently known. So there is still plenty of room for these methods to be extended and for new methods to be created in the pursuit of evaluating even more complex functions!

## Acknowledgements

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## References

[1] Alexandr Garbali, Jan de Gier, William Mead, and Michael Wheeler. Symmetric functions from the six-vertex model in half-space, 2024.

[2] Greg Kuperberg. Symmetry classes of alternating-sign matrices under one roof. 2001.

## What Really Counts

For the curious reader here are the first few numbers in the enumeration of $m\times m$ ASMs:

$$1, 1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, 129534272700, 31095744852375, 12611311859677500, \ldots$$

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