Bloch analysis of wave frequencies in a periodic elastic medium

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Introduction

Bloch's theorem has far ranging applications in quantum mechanics and continuum mechanics. It essentially states that in a periodic medium, wave solutions have a quasi-periodic solution.

In this paper we use Bloch's theorem to analyse the frequencies of a plane wave propagating through a periodic elastic medium with stiff circular inclusions with finite element methods.

Bloch's theorem

For our particular case with a periodic medium in the x-direction, we can write Bloch's theorem as such: Given a displacement field for a wave $\boldsymbol{\xi}(\boldsymbol{x}, t)$, the solution to the wave equation has the form

$$\boldsymbol{\xi}(\boldsymbol{x}, t) = \tilde{\boldsymbol{\xi}}(\boldsymbol{x})e^{i(\beta x - \omega t)} \tag{1}$$

Weak form of elasticity equation

In order to sole for $\tilde{\boldsymbol{\xi}}$ using finite element methods, we must multiply by a test function **v** and integrate to put the equation in 'weak form' (turning the differential equation into one involving integrals, optimal for numerical methods).

Define:

$$\varepsilon(v) = \frac{1}{2}(\nabla v + (\nabla v)^T)$$

We begin with the following equation for elastic mediums:

$$\nabla \cdot \boldsymbol{\sigma} = \rho \ddot{\boldsymbol{\xi}} \tag{2}$$

$$\text{where } \boldsymbol{\sigma} = \lambda(\nabla \cdot \xi)\boldsymbol{I} + 2\mu\varepsilon(\xi) \tag{3}$$

By bloch, for a lxh unit cell we can assume the solution takes the form

$$\boldsymbol{\xi}(\boldsymbol{x}, t; \beta) = \tilde{\boldsymbol{\xi}}(\boldsymbol{x}; \beta)e^{i(\beta x - \omega t)} \tag{4}$$

Note that $\nabla\boldsymbol{\xi} = e^{i\beta x}(\nabla\tilde{\boldsymbol{\xi}} + i\boldsymbol{k^T} \otimes \tilde{\xi})$ where $\boldsymbol{k^T} = \begin{bmatrix}\beta \\ 0\end{bmatrix}$.

$\therefore$ we can say $\nabla_\beta = \begin{bmatrix}\partial_x + i\beta \\ \partial_y\end{bmatrix}$ Substituting into (2) we get:

$$\nabla_\beta \sigma(\tilde{\boldsymbol{\xi}}) = -\omega^2 \rho \boldsymbol{\xi}$$

Multiplying by test function **v** and integrating using variational principles gives us

$$\int_\Omega \overline{S(\nabla_\beta \boldsymbol{v})} : (\lambda(\nabla_\beta \cdot \boldsymbol{\xi})\boldsymbol{I} + 2\mu\varepsilon_\beta(\boldsymbol{\xi}))d\Omega = \omega^2 \int_\Omega \rho(\overline{\boldsymbol{v}} \cdot \boldsymbol{\xi})d\Omega \tag{5}$$

Where S(M) is the symmetric part of the matrix M. We solve the eigenvalue problem with finite element methods, setting periodic boundary conditions $\tilde{\boldsymbol{\xi}}(0, y) = \tilde{\boldsymbol{\xi}}(L, yt)$

Dispersion relation and wave velocity

Starting with equation (2), we can apply Helmholtz decomposition letting

$$\xi_x = \frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial y}$$

$$\xi_y = \frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial x}$$

Decoupling orthogonal components gives us

$$\frac{\partial^2\phi}{\partial t^2} = c_l^2\nabla^2\phi \tag{6}$$

$$\frac{\partial^2\psi}{\partial t^2} = c_t^2\nabla^2\psi \tag{7}$$

Where $c_l^2 = \frac{\lambda+2\mu}{\rho}$ and $c_t^2 = \frac{\mu}{\rho}$. These are the wave speeds.

Now we can assume that $\phi = \tilde{\phi}(y)e^{i(kx-\omega t)}$ and $\psi = \tilde{\psi}(y)e^{i(kx-\omega t)}$. By substituting into (6) we get

$$\frac{d^2\tilde{\phi}}{dy^2} - \left(k^2 - \frac{\omega^2}{c_l^2}\right)\tilde{\phi} = 0$$

Giving us the longitudinal wave number defined by $a_l^2 = k^2 - \frac{\omega^2}{c_l^2}$.

Similarly, from (7) we can get the transverse wave number $a_t^2 = k^2 - \frac{\omega^2}{c_t^2}$

Finally, by substituting the Bloch ansatz we can see that for low frequency bands, after making suitable cancellations, the Bloch wavenumber β=k.

Finite element analysis for uniform elastic cell

Using the SLEPc library (Scalable Library for Eigenvalue Problem Computations), we solve (5) with periodic boundary conditions over a lxh unit cell, $\tilde{\boldsymbol{\xi}}(0, y) = \tilde{\boldsymbol{\xi}}(L, y)$.

We loop over different values of the bloch wavelength, β, and find the frequencies which support wave solutions.

We use $l = h = 2.35mm$, and first solve with an elastic medium of uniform Young's Modulus $E = 5$.

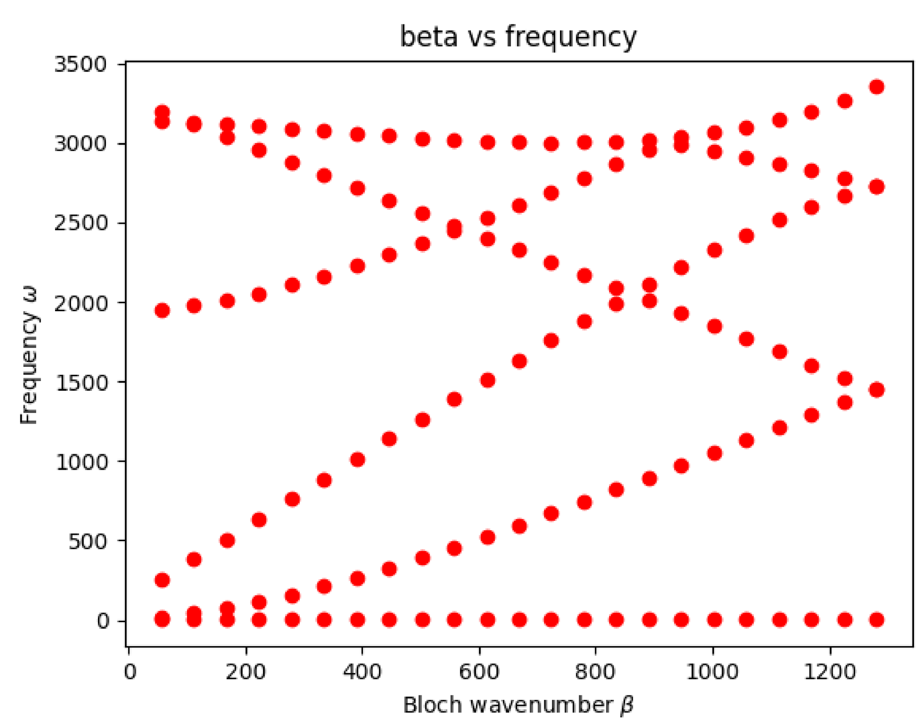

Figure 1. Beta vs frequency for uniform elastic cell

Here is an example solution with $\beta = \pi/3$ and $\omega = 266Hz$

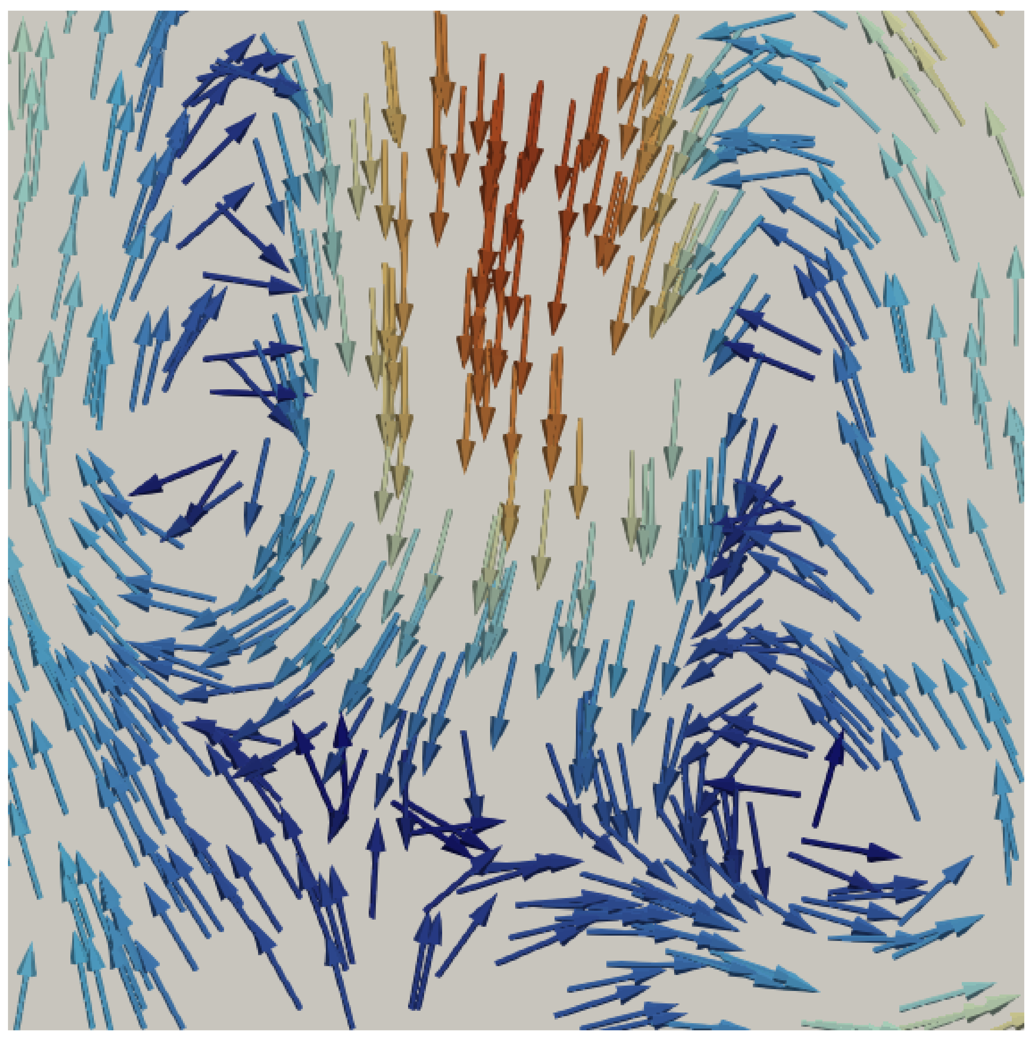

Figure 2. Example of a solution vector field

Analysis with circular inclusion

Now we insert a stiff circular inclusion with Young's Modulus $E = 15$ and radius $r = 0.47$mm into the center of the cell. After performing the same analysis, we get the results

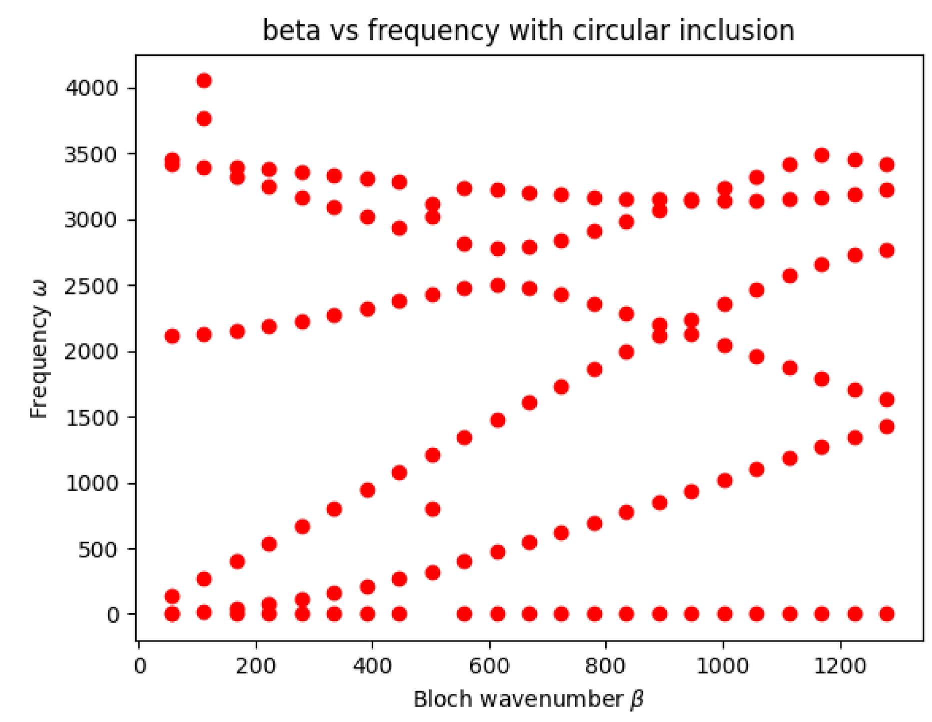

Figure 3. Beta vs frequency with stiff circular inclusion

There is some random error due to the resolution of the finite element mesh imposed by computing constraints. One can see that in the top 3 bands, the frequencies get altered, generating a band gap in these bands.

Future analysis can be improved with a higher resolution mesh supported by more computing power, and investigating the effect of different geometries on supported frequencies.

References

[1] Fabbiane et al. (2025). Phononic compliant surfaces for the suppression of travelling-wave flutter instabilities in boundary-layer flows. Journal of Fluid Mechanics

[2] Hussein, M. I. (2009). Reduced bloch mode expansion for periodic media band structure calculations.

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