Algebraic structures and geometric realisations of virtual and welded knots

Todd McGinley   Supervised by Arunima Ray and Marcy Robertson

THE UNIVERSITY OF MELBOURNE

Virtual knots

A **classical knot** is an embedding of a circle in $\mathbb{R}^3$, often represented by a knot diagram with over and under crossing information at each double point.

A **Gauss diagram** consists of an oriented circle with pairs of points joined by signed arrows. Each crossing in a knot diagram gives an arrow directed from the under–crossing to the over–crossing and the sign by the sign of the crossing.

Not every Gauss diagram comes from a classical knot: the ordering of crossings may force extra intersections. A **virtual knot** is a knot diagram that allows an additional type of crossing, called virtual crossings, which carries no over/under information, allowing realisations of all Gauss diagrams.

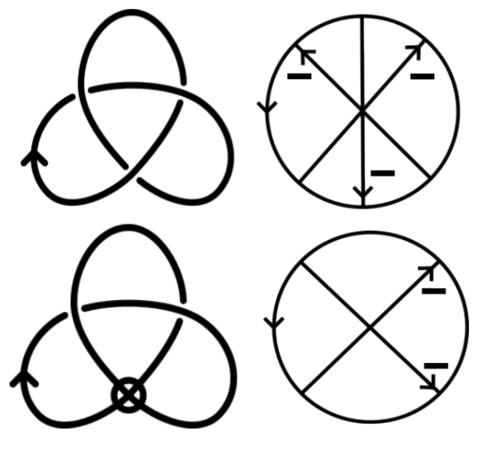

Figure 1. A classical and virtual knot diagram (left) and their corresponding Gauss diagrams (right)

Virtual knots are considered up to Reidemeister moves and planar isotopies. Two virtual knots are equivalent if they are related by a sequence of these moves/isotopies. The virtual Reidemeister moves are shown in Figure 2. The classical moves are the virtual moves 1-3 with virtual crossings replaced with classical crossings with opposite signs in move 2 and a common sign in move 3.

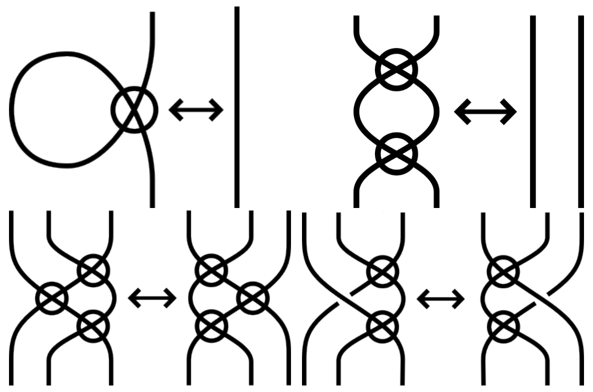

Figure 2. Virtual moves 1-3 and the mixed move

Virtual knots can be modeled as knots embedded in thickened surfaces, considered up to addition/removal of a handle. Virtual crossings arise from distinct segments of the surface projecting to the same region of the plane.

A **Welded knot** is a virtual knot diagram, but we allow over-crossings to pass over virtual crossings, we don't allow the same for under-crossings as then every knot would become trivial.

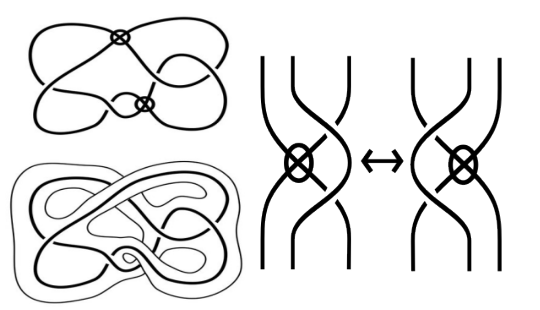

Figure 3. A virtual knot and a realisation of it living inside a thickened surface (left). Source J. Purcell Twitter, the welded move (right)

Tangles

A **Tangle** is an embedding of a collection of oriented intervals and circles into a box, where the endpoints of the intervals must lie evenly distributed along the middle of the top or bottom of the box. Equivalence of tangle diagrams can be checked using the Reidemeister moves for knots, however you cannot move endpoints.

A classical tangle refers to any tangle without virtual crossing. We also have framed tangles which are classical tangles with a modified Reidemeister 1 move (Figure 4). Virtual tangles can contain virtual crossings, but do not need to have them.

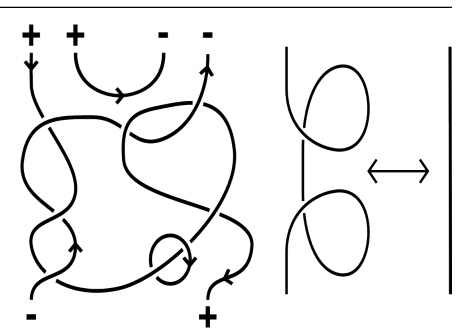

Figure 4. A classical tangle (left) and the modified Reidemeister 1 move (right)

Outline

In column 2 we will introduce some category theory that allows us to easily see structure within classical/virtual tangles and gives us some very powerful invariants.

In column 3 we discuss knotted tori in S^4 and explain how a special class of knotted tori corresponds to welded knots.

Abstract nonsense

A **category** $\mathcal{C}$ consists of objects and morphisms between them, with associative composition and identity morphisms. A **functor** is a map between categories preserving identities and composition. A map is **natural** if we can move morphisms through it.

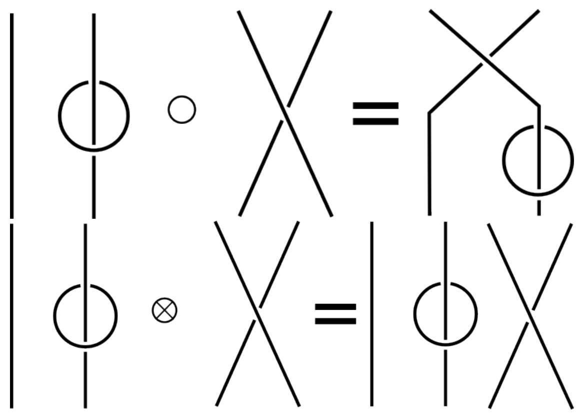

Figure 5. Tangle composition and tensor product (orientation dropped)

A category is **monoidal** if it is equipped with a tensor product $\otimes$ and a unit object I. If associativity and unit laws hold by equality, it's called **strict**, otherwise they hold up to natural isomorphism. A functor is **monoidal** if it preserves this.

A monoidal category is **braided** if there are natural isomorphisms $\beta_{X,Y} : X \otimes Y \to Y \otimes X$ satisfying the hexagon identities:
$\beta_{X,Y\otimes Z} = (id_Y \otimes \beta_{X,Z}) \circ (\beta_{X,Y} \otimes id_Z)$
$\beta_{X\otimes Y,Z} = (\beta_{X,Z} \otimes id_Y) \circ (id_X \otimes \beta_{Y,Z})$
If $\beta_{Y,X} \circ \beta_{X,Y} = id$, the category is **symmetric**.

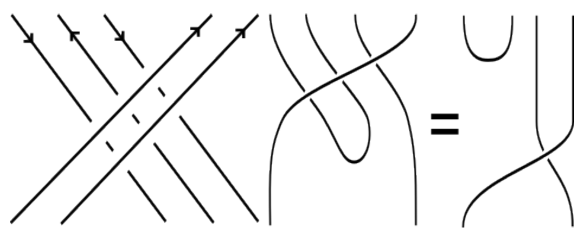

Figure 6. Braiding in tangles and an example of it's naturality (orientation dropped)

Classical and virtual tangles form strict monoidal categories, objects are words of "+"/"-" forced by orientation (Figure 7). Classical tangles admit a braiding via under-crossings (Figure 6), while virtual tangles yield a symmetric structure by using virtual crossings.

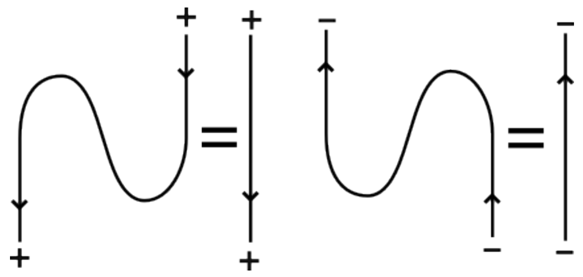

Figure 7. Rigidity laws in the tangle category

A monoidal category is **rigid** if every object, A, has a dual object A^* and there exists maps
$\mathrm{ev}_A : A \oplus A^* \to I, \mathrm{coev}_A : I \to A^* \otimes A$ such that
$(\mathrm{ev}_A \otimes id_A) \circ (id_A \otimes \mathrm{coev}_A) = id_A$
$(\mathrm{id}_{A^*} \otimes \mathrm{ev}_A) \circ (\mathrm{coev}_A \otimes id_{A^*}) = id_A$
We can also define a dual on maps, for $f : A \to B$ $f^* : B^* \to A^*, f^* = (id_{A^*} \otimes \mathrm{ev}_B) \circ (id_{A^*} \otimes f \otimes id_{B^*}) \circ (\mathrm{coev}_A \otimes id_{B^*})$

A rigid braided monoidal category is **ribbon** if there exists a family of natural endomorphisms θ_X called twists that satisfy
$\theta_{X\otimes Y} = \beta_{Y,X} \circ \beta_{X,Y} \circ (\theta_X \otimes \theta_Y), \theta_I = id_I, \theta_{X^*} = \theta_X^*$

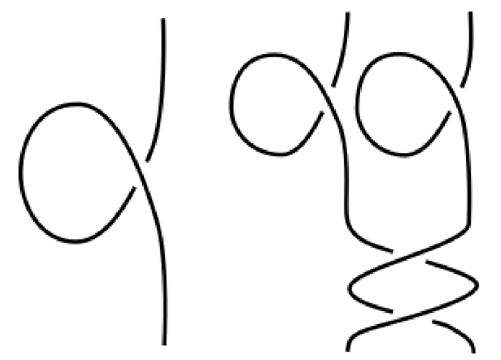

Figure 8. Twists on framed tangles with one/two strands (orientation dropped), making framed tangles ribbon

Categorical structure of tangles

Let $\mathcal{T}$ be the category of framed tangles and $v\mathcal{T}$ the category of irrotational virtual tangles, disallowing virtual move 1.

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\iota} & v\mathcal{T} \\ \downarrow F_V & & \downarrow G_V \\ \mathcal{C} & \xrightarrow{G} & \mathcal{S} \end{array}$$

Figure 9.

Theorem (Shum, Reshetikhin, Turaev): let $\mathcal{R}$ be a ribbon category, for any object $A \in \mathcal{R}$, there exists a unique strict monoidal functor $F_A : \mathcal{T} \to \mathcal{R}$ preserving the ribbon structure, with $F_A(+) = A, F_A(-) = A^*$

Theorem (Brochier): let $\mathcal{R}, V, F_A$ be as above, $\mathcal{S}$ a strict symmetric monoidal category and $G : \mathcal{R} \to \mathcal{S}$ a strong monoidal functor. Then there exists a unique strict symmetric monoidal functor $G_A : v\mathcal{T} \to \mathcal{S}$ such that $G_A(+) = G(A)$ and Figure 9 commutes.

Now we have very powerful invariants for both framed and virtual tangles given by the ribbon category of finite dimensional representations of quantum groups.

Ribbon knots and knotted tori

A **ribbon knot** is a knot that bounds a disk that is allowed to pass through itself along intervals (Figure 10)

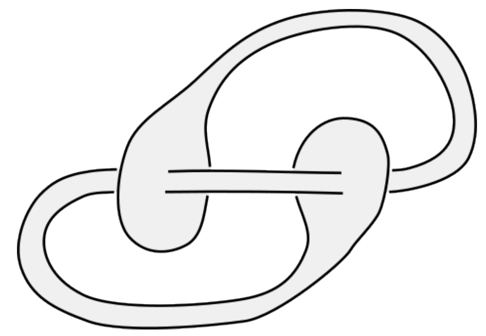

Figure 10. A ribbon knot with disk

A **knotted torus** is an embedding of the torus T into S^4, it is said to be **ribbon** if it bounds a solid torus with two orientations, a 3-dimensional orientation and an orientation on its core circle, this solid torus can intersect itself along disks, much like a ribbon knot, but every component is one dimension larger.

A **broken torus diagram** is a torus immersed in $\mathbb{R}^3$ that intersects itself along a finite number of circles. The over/under information of these intersections is indicated on the diagram by erasing a small neighborhood of the lower sheet. The diagram is symmetric if it is locally homeomorphic to one of the images in Figure 11.

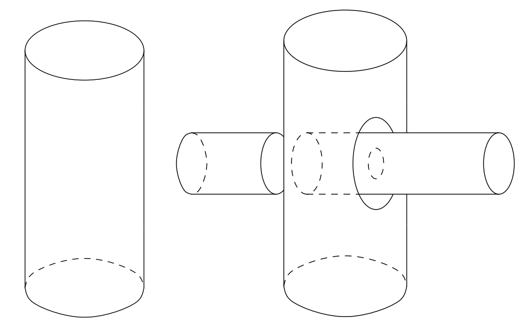

Figure 11. Pieces of symmetric broken torus diagrams

Symmetric broken torus diagrams are considered up to their own Reidemeister moves. There are the 3 classical Reidemeister moves with crossings replaced with the symmetric crossing and its mirror image, and one more move allowing the interchange of tubes (Figure 12)

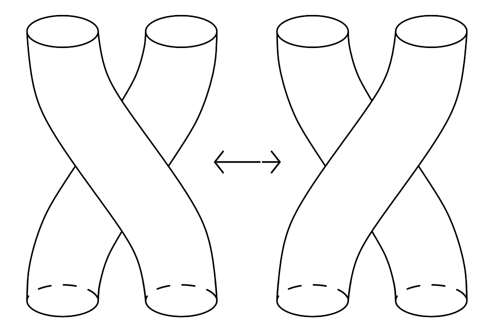

Figure 12. The interchange move

The tube map

There is a one-to-one correspondence between symmetric broken torus diagrams and ribbon torus knots. Further we can define a map from welded knots to symmetric broken torus diagrams, called the **tube map**, by inflating the circle to a torus and using Figure 13 to determine what happens at crossings.

It's known the tube map is not bijective as there exists a reversal on welded knots that doesn't affect the image of the map. It's an open problem if quotienting out by this reversal gives a bijection.

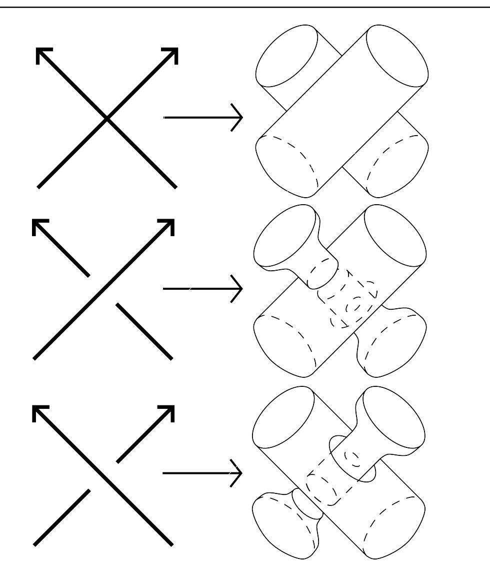

Figure 13. The tube map on crossings

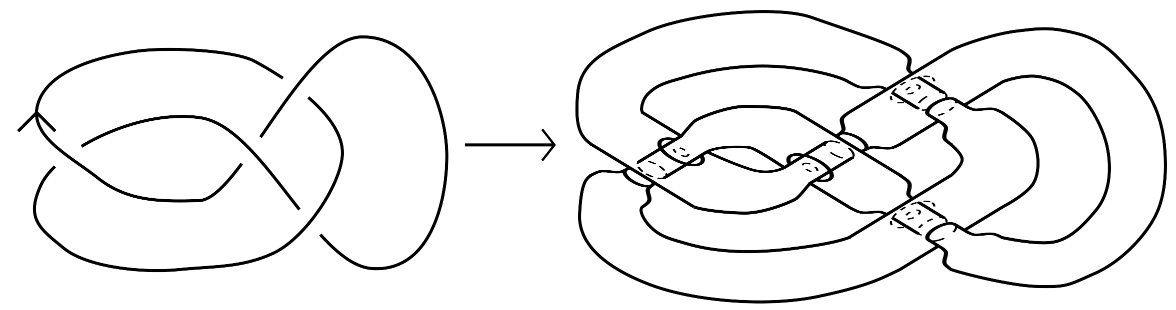

Figure 14. A broken torus diagram of a non-trivial ribbon torus knot, given by the welded tube map

References

[1] S. Chmutov, S. Duzhin, and J. Mostovoy. *Introduction to Vassiliev Knot Invariants*. Cambridge University Press, 2012.

[2] Adrien Brochier. Virtual tangles and fiber functors. *Journal of Knot Theory and Its Ramifications*, 28(07), 2019.

[3] Benjamin Audoux. On the welded tube map. In *Knot theory and its applications*, volume 670 of *Contemp. Math.*, pages 261–284. Amer. Math. Soc., Providence, RI, 2016. Figures 11-14 use material from this paper.

tmcginley@student.unimelb.edu.au