Minimal directed Steiner networks

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Background

How do you connect a collection of cities using the least amount of road as possible? This optimisation problem is the **Euclidean Steiner tree problem**, and the solution involves the use of 'intersections', or Steiner points. In a minimal network

- All Steiner points have degree 3, with edges ('roads') meeting at 120°
- Terminals ('cities') have degree at most 3
- There are no cycles, and
- There are at most $n-2$ Steiner points for an n terminal network

A shortest tree for points on the vertices of a square is shown in Fig. 1.

There is an algorithm known as **GeoSteiner** which is able to solve this problem efficiently, taking advantage of geometric properties of the minimal trees to cut down on the work required [2]. It

- Grows potential components of the final solution
- Excludes non-minimal components for efficiency
- Connects the terminals using components to minimise total length

Minimal square networks

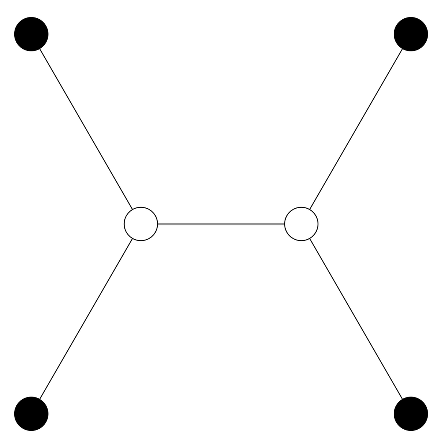

Fig. 1: Undirected

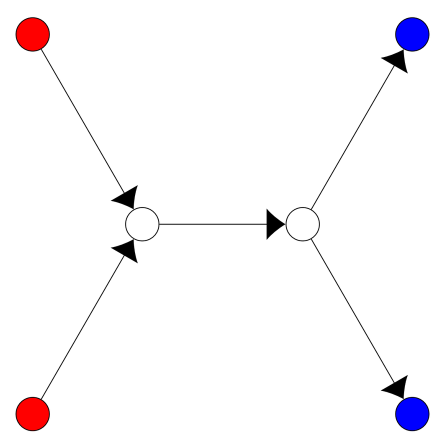

Fig. 2: Directed

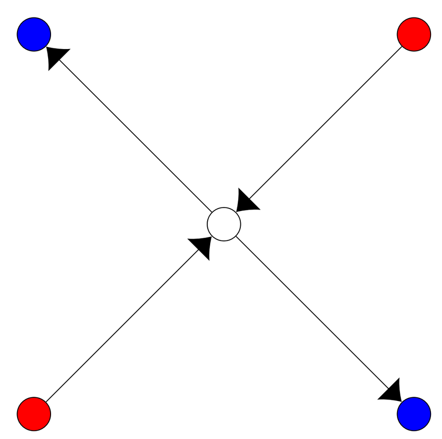

Fig. 3: Directed, terminal swap

Directed Steiner networks

A variation of the Euclidean Steiner tree problem is the directed problem, in which the terminals are classified as sources or sinks, and every source must have a directed path to every sink. Whilst some minimal directed networks are similar to undirected equivalents, others differ (see Fig. 2 and Fig. 3). In these minimal networks,

- Steiner points can have degree 3, 4, 5, or 6 [3]
- Cycles may be present, including cycles of Steiner points [1]
- There are at most $52n$ Steiner points, for an n terminal network [1]
- There is no known algorithm for their construction

We have worked to determine what must be done to develop such an algorithm.

Location of Steiner points

An important part of finding a minimal network will be a procedure to find the location of Steiner points given a topology – some terminals, some number of Steiner points, and their connections.

- The **Melzak-Hwang algorithm** [2] is used in the undirected case
- It can be adapted for many directed cases
- A notable exception: networks with a cycle of degree 3 Steiner points
- Some degree 3 Steiner cycles can be ignored, as they can be expanded/contracted and removed (see Fig. 7)
- Not all degree 3 Steiner cycles share this property

Degree 3 Steiner cycle geometry

We wish to know when the problematic cycles can be ignored, and if we can precisely locate the Steiner points of the remainder. Thus we analyse the geometry of these cycles, based on the points around them. We find

- External points restrict the location of the Steiner points (Fig. 4)
- Steiner point geometry forces the angles at p_1 and p_2 to remain fixed
- The upper outer arc is p_1 as θ varies from 0° to 120° (Fig. 5)
- The upper inner arc is p_2 as α varies from 0° to 60° (Fig. 6)

For all θ and α in those ranges, the Steiner points have a range of possible locations: up and down the equilateral triangle in Fig. 5, and along the rectangle in Fig. 6. Intersections of the restrictive regions around a cycle will indicate the location of Steiner points, and will be researched further.

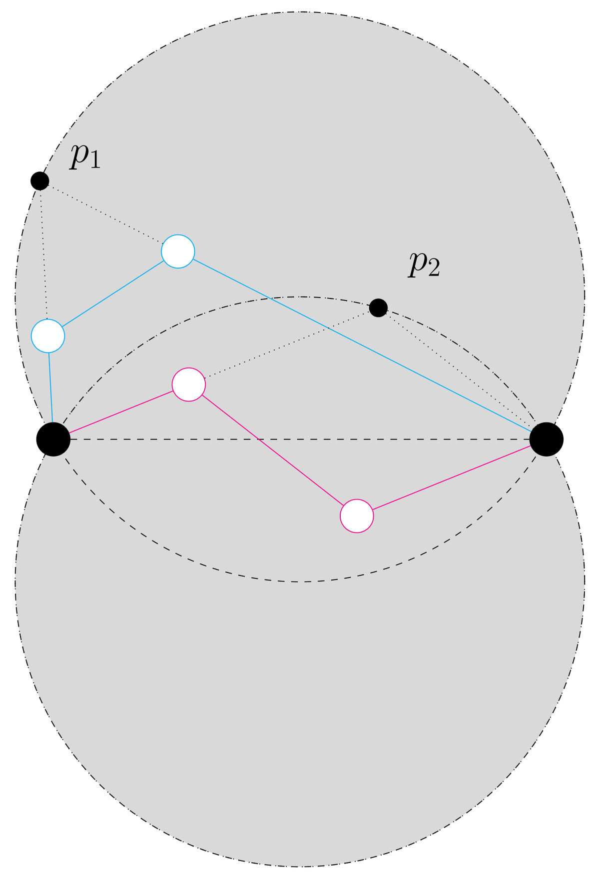

Fig. 4: Region where degree 3 Steiner cycle points could be located given adjacent external points, examples from Fig. 5 and Fig. 6 overlayed

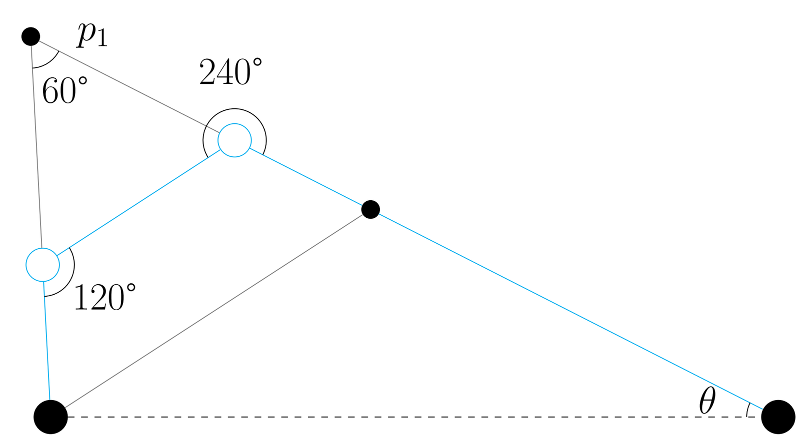

Fig. 5: External points on same side of cycle

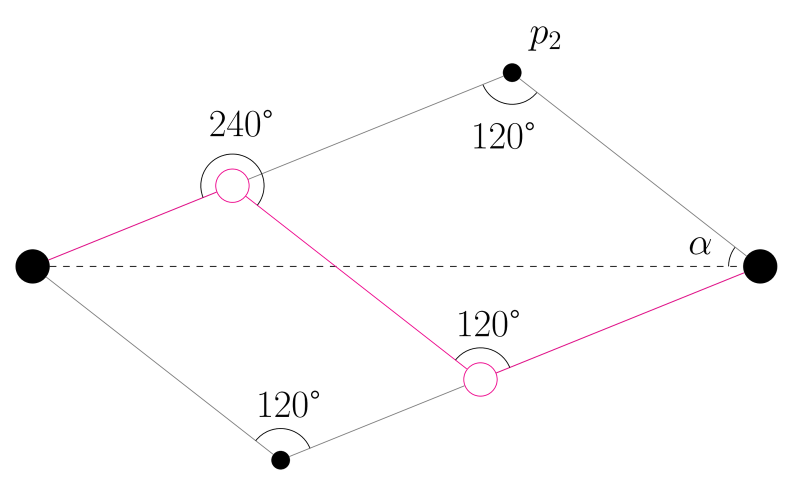

Fig. 6: External points on alternate sides of cycle

Flexible Steiner cycle

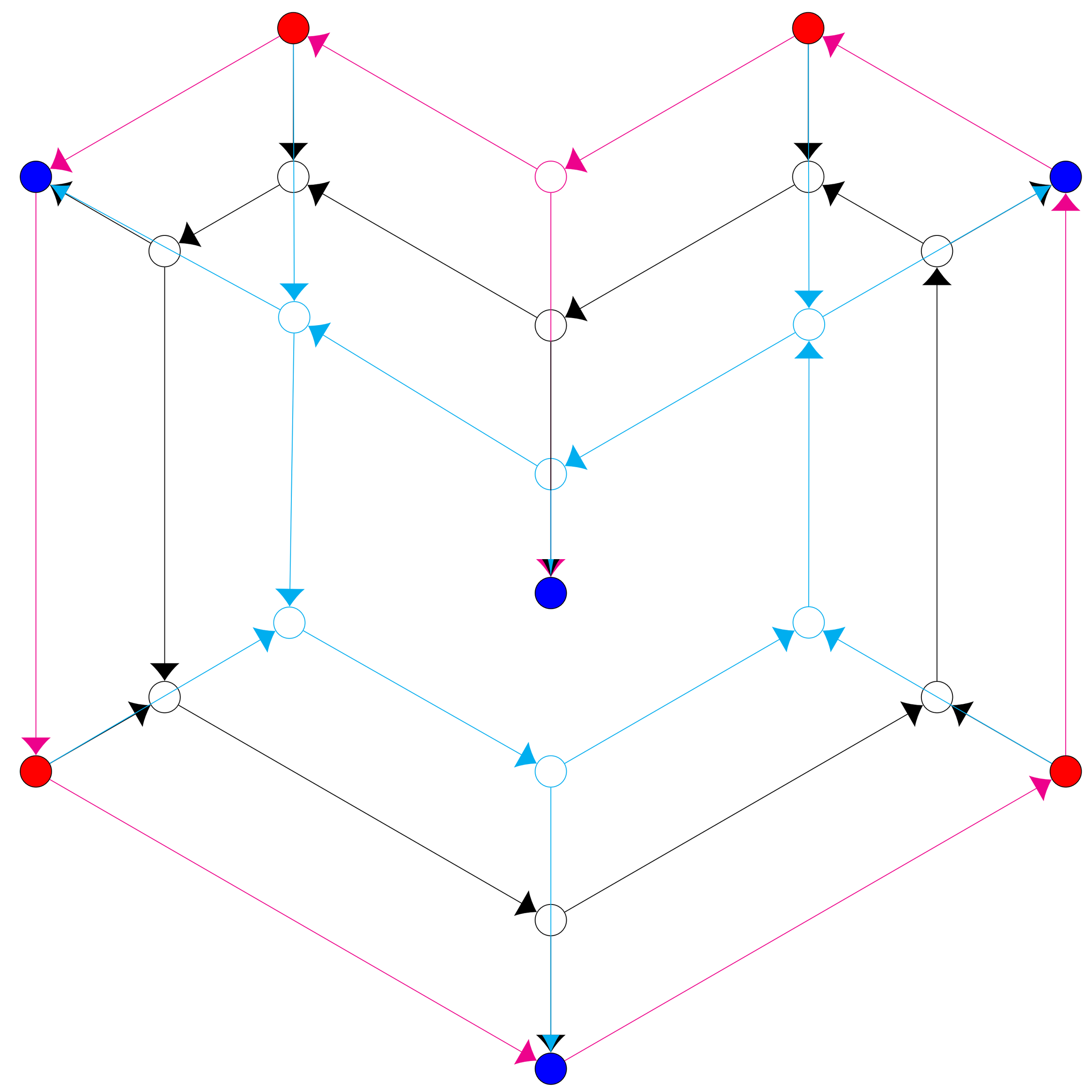

Fig. 7: Equal length networks, obtained by expanding/contracting the degree 3 Steiner cycle (black). The degree 3 Steiner cycle degenerates at the extremes, leaving behind a favourable cycle for an adaptation of the Melzak-Hwang algorithm

Acknowledgements

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References

[1] Manuel Alfaro. Existence of shortest directed networks in $\mathbf{R}^2$. *Pacific Journal of Mathematics*, 167(2):201–214, 1995.

[2] Marcus Brazil. *Optimal interconnection trees in the plane.* Springer, 2015.

[3] Alastair Maxwell and Konrad J Swanepoel. Shortest directed networks in the plane. *Graphs and Combinatorics*, 36(5):1457–1475, 2020.