Tannaka Duality for Groups

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Introduction

A natural way to study groups is to examine their actions. This enables us to encode group elements as automorphisms of the target structure. In this way, actions may help us deduce information about the underlying group. Of course, actions are necessarily only a shadow of the original group. So, we ask: given a group and its category of representations, how can we recover the group itself? In this poster, we explore Tannaka Duality for groups for the specific cases of finite dimensional vector spaces over a field k, $Rep_k(G)$ and sets, $Rep_{Set}(G)$. Our aim is to establish the relationship between an arbitrary group and the automorphism group of the forgetful functor associated to $Rep_{Set}(G)$ and $Rep_k(G)$ respectively.

Reconstructing G from the forgetful functor F_{Set}

To recover G, we require several lemmas. Given categories C, D and functors $E, G : C \to D$ such that η is a natural isomorphism $E \Rightarrow G$, then we can induce a monoid isomorphism $End(E) \cong End(G)$ by the following map

$$\varphi : End(E) \to End(G)$$
$$\alpha \mapsto \eta \circ \alpha \circ \eta^{-1}$$

By applying the Yoneda lemma, we can deduce two important lemmas. Firstly, $Hom(Hom(\bullet, -), -) \cong F$ as functors. Secondly, for a locally small category C and $X \in C$, then $Nat(Hom(X, -), Hom(X, -)) \cong Hom^{op}(X, X)$ as a monoid isomorphism. Finally, we can see that

$$\begin{aligned} End(F_{Set}) &\cong End(Nat(Hom(\bullet, -), -)) \\ &\cong Hom^{op}(Hom(\bullet, -), Hom(\bullet, -)) \\ &\cong Hom(\bullet, \bullet) \\ &\cong G \end{aligned}$$

Indeed, since $End(F_{Set}) = Aut(F_{Set})$ then $G \cong Aut(F_{Set})$.

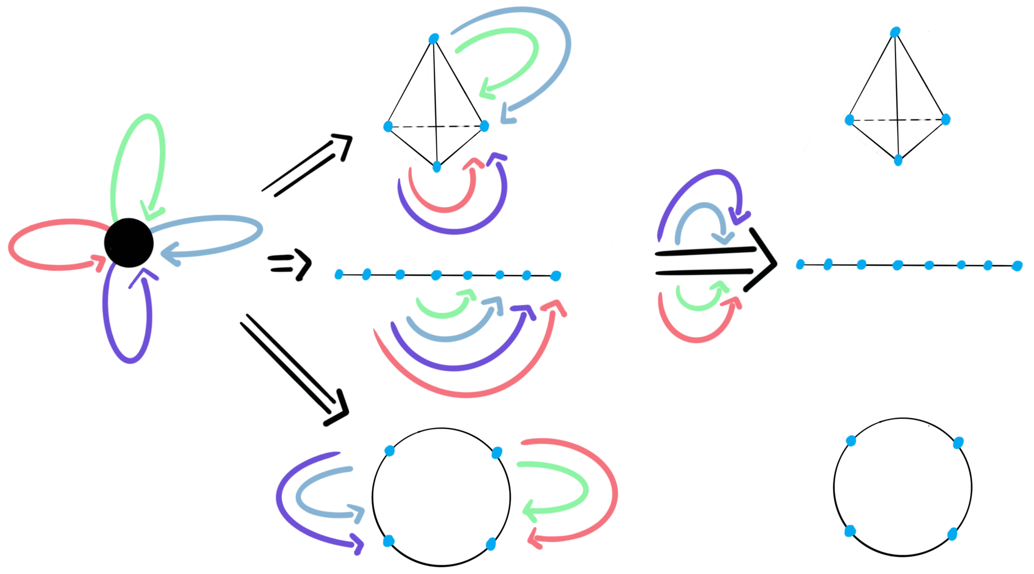

Figure 2: Visualising $Rep_{Set}(G)$ and $F_{Set} : Rep_k Set \to Set$

Reconstructing G from the forgetful functor F_{Vec}

For this case we impose the restriction that G is finite. Define the map

$$\gamma : G \to Aut(F_{Vec})$$
$$g \mapsto (\rho(g)_{(V,\rho)})_{(V,\rho)}$$

Consider the representation $(k[G], \tau)$ where $k[G]$ is the group algebra and τ is left multiplication. Our map is injective since if $\tau(g) = id_{k[G]}$ then

$$gh = h \forall h \in G \implies g = e$$

Indeed, $k[G]$ is special since we can show that any $\eta \in End(F_{Vec})$ is completely determined by the component $\eta_{(k[G],\tau)}$. Assume that $\eta_{(k[G],\tau)} = \varphi_{(k[G],\tau)}$, then for arbitrary $(V, \rho), v \in V$, define

$$\lambda_v : k[G] \to V$$
$$1 \cdot e_g = 1 \mapsto v$$
$$\sum_{i=1}^{n} a_i g_i \mapsto \sum_{i=1}^{n} a_i \rho(g_i) v$$

Then $\lambda_v \in Hom((k[G], \tau), (V, \rho))$ ensures the following is commutative giving the desired result.

$$\begin{tikzcd} k[G] \arrow[r, "\lambda_v"] \arrow[d, shift right, "\eta_{(k[G],\tau)}"'] \arrow[d, shift left, "\varphi_{(k[G],\tau)}"] & V \arrow[d, shift right, "\eta_{(V,\rho)}"'] \arrow[d, shift left, "\varphi_{(V,\rho)}"] \\ k[G] \arrow[r, "\lambda_v"'] & V \end{tikzcd}$$

Furthermore, right multiplication by an element $s \in G$ defines a morphism

$$R_s : (k[G], \tau) \to (k[G], \tau)$$

which enforces that $\eta_{(k[G],\tau)}$ is left multiplication by some $\sum_{i=1}^{n} a_i g_i \in k[G]$. Therefore, any natural isomorphism amounts to left multiplication by an element of the multiplicative group of $k[G]$. Finally, if $\eta \in Aut^{\otimes}(F_{Vec})$, then for arbitrary representations $(V, \rho_1), (W, \rho_2)$, imposing the equality

$$\eta_{(V,\rho_1)\otimes(W,\rho_2)} = \eta_{(V,\rho_1)} \otimes \eta_{(W,\rho_2)}$$

implies

$$\sum_{i=1}^{n} a_i \rho_1(g_i) \otimes \sum_{i=1}^{n} a_i \rho_2(g_i) = \sum_{i=1}^{n} a_i \rho_1 \otimes \rho_2(g_i)$$

This equality can only hold if $a_j = 1$ for some fixed $j \in \{1, ..., n\}$ and $a_i = 0$ for all $i \neq j$. By the definition of tensoring k-linear representations, it follows that $\eta \in Aut^{\otimes}(F)$ if and only if $\eta = (\rho(g))_{(V,\rho)}$ for some $g \in G$. Therefore, γ is surjective and we can conclude

$$G \cong Aut^{\otimes}(F_{Vec})$$

The category $Rep_{Set}(G)$

Consider a group G as the one object category $\mathbf{B}G$ such that for all $g, h \in G$ $g \circ h = gh$.

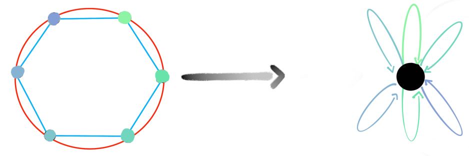

Figure 1: The group $\mathbb{Z}/6\mathbb{Z}$ considered as a category.

A functor $U : \mathbf{B}G \to Set$ maps $\bullet$ to some $X \in Set$ and $g \in End(\bullet)$ to a function $f_g : X \to X$. By the functorality of U, we have $U(e) = id_X$ and for all $x \in X, g, h \in End(\bullet), U(g \circ h)(x) = U(g) \circ U(h)(x)$. By setting $g \cdot x = U(g)(x)$, it is clear that U defines a group action. Therefore, define Rep_{Set} as the functor category $Hom(\mathbf{B}G, Set)$. This category is equipped with a forgetful functor $F_{Set} : Hom(\mathbf{B}G, Set) \to Set$ which sends $U \in Hom(\mathbf{B}G, Set)$ to $U(\bullet)$ and acts trivially on morphisms.

The category $Rep_k(G)$

Consider the category $Rep_k(G)$ which has as objects pairs (V, ρ) where V is a finite-dimensional vector space and $\rho : G \to Aut(V)$ is a group homomorphism. A morphism

$$\phi : (V, \rho) \to (W, \theta)$$

is a linear map such that for all $s \in G, \phi \circ \rho(s) = \theta \circ \phi(s)$. $Rep_k(G)$ is also equipped with a forgetful functor, F_{Vec}. To establish a similar result for $Rep_k(G)$, we must note that it has an additional monoidal structure induced by the tensor product. We can ask: given $\eta \in Aut(F_{Vec})$, does it preserve tensor products? We define the subgroup $Aut^{\otimes}(F_{Vec}) \subset Aut(F_{Vec})$ such that

$$\eta \in Aut^{\otimes}(F_{Vec}) \iff \eta_{(V,\rho)\otimes(W,\theta)} = \eta_{(V,\rho)} \otimes \eta_{(W,\theta)}$$

for all $(V, \rho), (W, \theta) \in Rep_k(G)$.

A closer look at $k[G]$

Given $c \in k$, $\alpha, \beta \in End(F_{Vec})$, define

$$\alpha + \beta := (\alpha_{(V,\rho)} + \beta_{(V,\rho)})_{(V,\rho)}$$
$$c \cdot \beta := (c\beta_{(V,\rho)})_{(V,\rho)}$$

These maps give $End(F_{Vec})$ the structure of a vector space and defining $\alpha \cdot \beta := \alpha \circ \beta$ renders $End(F_{Vec})$ an associative algebra. Extending γ linearly defines an algebra homomorphism $\gamma : k[G] \to End(F)$. Indeed, it is possible to adapt the proof in the bottom left to deduce that $\{\tau(g_1), ..., \tau(g_n)\}$ is a basis for $End(F_{Vec})$, therefore $\gamma : k[G] \to End(F_{Vec})$ is an isomorphism of algebras.

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Full paper: https://jamesfkelly.github.io/

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