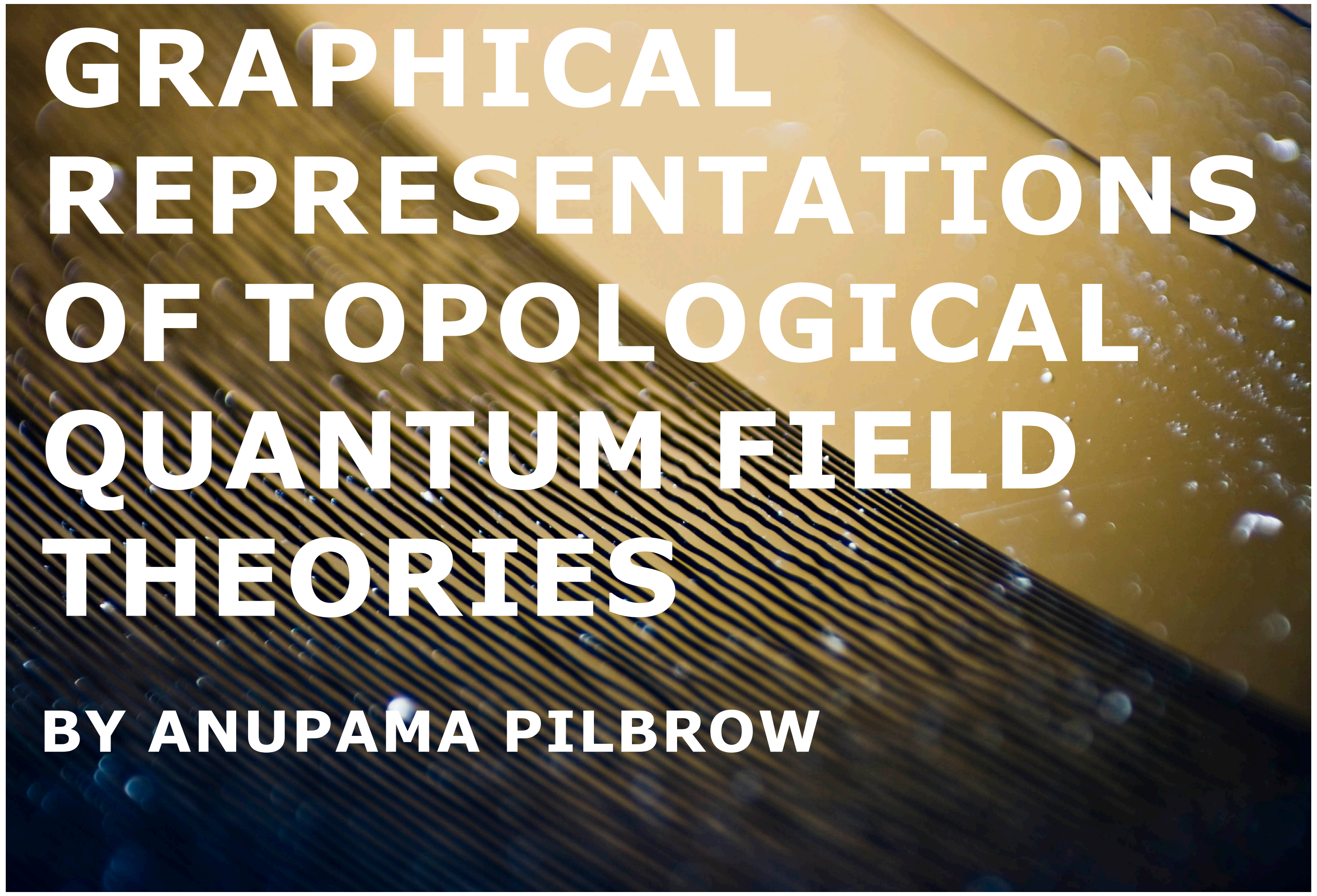

GRAPHICAL REPRESENTATIONS OF TOPOLOGICAL QUANTUM FIELD THEORIES

BY ANUPAMA PILBROW

OVERVIEW

We are interested in extracting information in a given topological quantum field theory (TQFT). Rather than dealing with difficult tensor product notation, we can consider the far more convenient graphical representations of TQFTs.

These graphical representations provide a good insight and a good point of comparison when exploring topological invariants, particularly through recursion relations.

In this project, we built up an understanding of TQFTs over four weeks until we were able to focus on some calculations to do with Eynard-Orantin symplectic invariants. We specifically considered some of the simpler Eynard-Orantin symplectic invariants and their corresponding graphical representations.

We mainly considered 2D TQFTs.

WHAT IS A TQFT?

We read some introductory works on TQFTs. Namely, *An Introduction to Topological Field Theory* by Ruth Lawrence. These works took an axiomatic approach in their definitions. We also drew diagrams to aid our understanding. We learned that 2D TQFTs and Frobenius algebras are equivalent. Swapping between the algebraic and the topological perspective proved helpful.

Definition A *2D TQFT* is a functor from the category of 2D cobordisms between 1D manifolds to the category of vector spaces.

Definition A *Frobenius algebra* is a finite-dimensional unital algebra equipped with a nondegenerate bilinear form, $\beta: V \otimes V \to \mathbb{C}$.

We defined a three-point function, $c: V \otimes V \otimes V \to \mathbb{C}$. *Figure 1* shows that β and c are associated with the "cylinder" and "pair-of-pants" .

KEY CONCEPTS

Idempotent Bases

An idempotent basis is composed of vectors satisfying

$$e_i \cdot e_i = e_i$$
$$e_i \cdot e_j = 0$$
$$\text{where } \sum_i e_i = 1$$

We were able to find idempotent bases for the complex vector spaces associated to the boundary components in a TQFT.

Graphical Representation

To avoid messy tensor notation, we represented calculations with diagrams. The more intuitive diagram of a 2D manifold (see *Figure 1*) gave way to simpler diagrams composed of trivalent vertices and edges (see *Figure 2*). A trivalent vertex was associated to each pair-of-pants and an edge to each cylinder.

There are a few ways of 'coloring in' these diagrams with basis vectors, one of which is shown in *Figure 2* for the vector space spanned by the vectors $\{e_0=(1,0), e_1=(0,1)\}$.

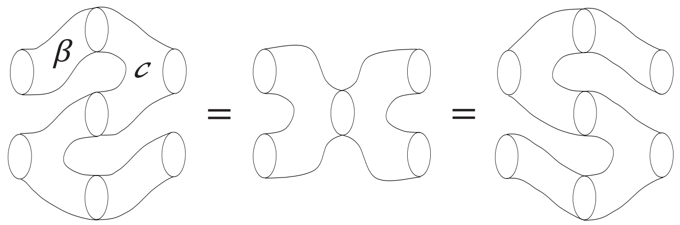

Figure 1
Diagrammatic representation of the Frobenius relation showing the bilinear form and the three-point function.

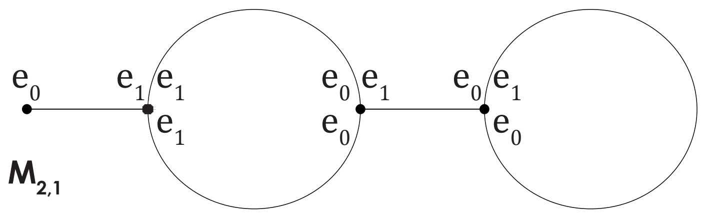

Figure 2
Non-zero coloring of genus 2 manifold with one boundary component. There are four possible non-zero colorings for this graph.

Graph Colorings and Weight

We want to calculate the weight of the graph in Figure 2 taking e_0 as input. Values for the bilinear form and the three-point function in this vector space are:

$$\beta(e_0, e_1) = \beta(e_1, e_0) = 1,$$
$$c(e_1, e_1, e_1) = 1 \text{ and}$$
$$c(e_1, e_0, e_0) = c(e_0, e_1, e_0) = c(e_1, e_0, e_0) = 1$$
and, otherwise, zero.

The weight of the graph is given by taking the sum of the values of each nonzero coloring.

$$Z(M_{2,1}) = \sum_{\text{colorings}} \beta^E \cdot c^V$$
for V = #vertices and E = #edges.

Towards Eynard-Orantin Invariants

The next step was to consider the idempotent basis. We expected to find that $Z(M_{g,n}) = 2^g$. We used our understanding of these graphs and colorings to get a feel for lower genus Eynard-Orantin invariants.

RESEARCH EXPERIENCE

I enjoyed my first opportunity to do research in mathematics. Learning how to read research papers and isolate relevant information while encountering unfamiliar terminology was very rewarding! I had fun doing some simple programming using Maple and learned a lot through conversations with Paul and Campbell.

I'd encourage any students interested in pursuing math to consider applying for a Vacation Scholarship. I had a great time and learned a lot—about TQFTs, and about what a future in math can look like!

ACKNOWLEDGEMENTS

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