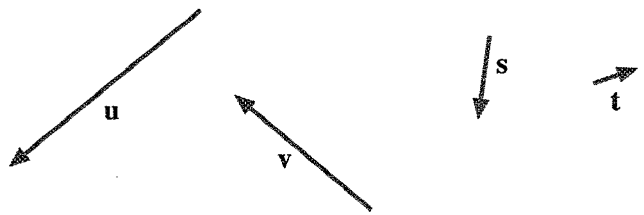
Topic 1: Vectors

There are many different types of quantifiable objects, for example, time, length, speed and height. We can quantify these things with a single (real) number which we call a *scalar*. Scalar quantities have *magnitude* only.

Some quantities, however, require more than just a scalar magnitude to be completely specified, for example, displacement, velocity and force. These quantities also have a *direction*. We call these quantites **vectors**. Vectors have magnitude and direction.

1.1 Vector notation

We can represent a vector diagramatically as a directed line segment



The length of the line represents the magnitude of the vector and the orientation of the line represents its direction.

There are several different ways of writing a vector; $\overrightarrow{AB}$ means the vector starting at the point A and ending at the point B. The notation $\mathbf{u}$ (this is a boldface u) is also used to indicate that the quantity you are using is a vector rather than a scalar. When we are writing a vector by hand we often use a symbol called a *tilde*, $\underset{\sim}{u}$.

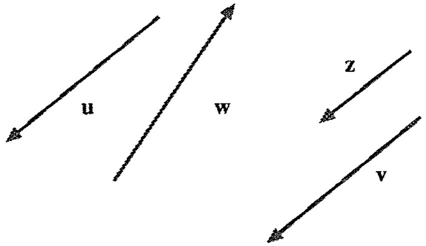
1.2 Vector algebra

Since we are no longer dealing with real numbers, we need to define what we mean by equality, addition, subtraction and so on, for vectors.

1.2.1 Equality of vectors

Since vectors have both magnitude and direction, to say that two vectors are equal means that they must have the same direction and the same magnitude.

Example:

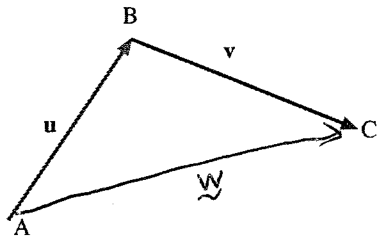


In the figure above we can make the following statements:

$$\mathbf{u} = \mathbf{v}$$
$$\mathbf{u} \neq \mathbf{w} \quad \text{(since they have different direction)}$$
$$\mathbf{u} \neq \mathbf{z} \quad \text{(since they have different magnitude).}$$

1.2.2 Addition of vectors

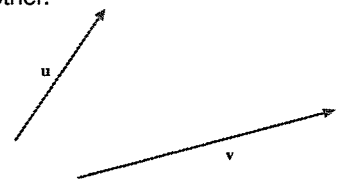
Suppose now that we have two vectors; the vector $\mathbf{u}$ which represents the vector starting at the point A and ending at the point B, and the vector $\mathbf{v}$ which represents the vector starting at the point B and ending at the point C.



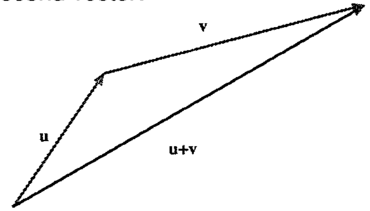
The overall effect can be represented by a new vector $\mathbf{w}$ which starts at the point A and ends at the point C. In this sense, we say that

$$\mathbf{w} = \mathbf{u} + \mathbf{v}.$$

In general, to add two vectors, take the tail of one vector and join it to the head of another.



The resultant vector starts at the tail of the first vector and ends at the head of the second vector:

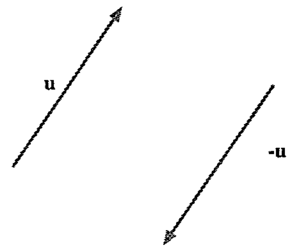


We call this resultant vector, the **sum** of the two vectors $\mathbf{u}$ and $\mathbf{v}$.

1.2.3 The zero vector

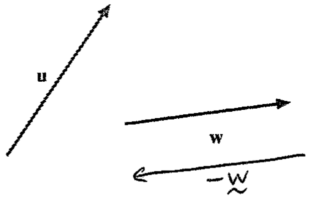
The zero vector is the vector that has zero length and no direction. It is denoted by $\mathbf{0}$.

1.2.4 The negative of a vector



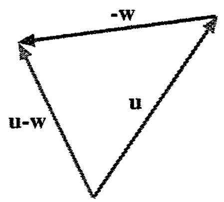
If $\mathbf{u}$ is the vector from A to B then $-\mathbf{u}$ is simply the vector from B to A. So the negative of a vector has the same magnitude but the opposite direction of the original vector.

We can subtract the vector $\mathbf{w}$ from the vector $\mathbf{u}$ by adding the negative of $\mathbf{w}$ to $\mathbf{u}$.

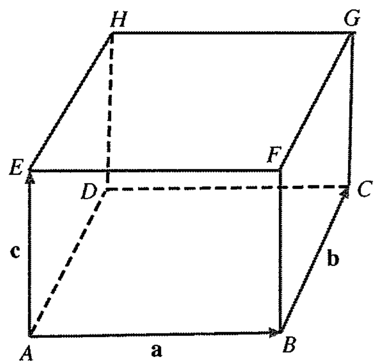


That is, we reverse the direction of $\mathbf{w}$ and then add:

$$\mathbf{u} - \mathbf{w} = \mathbf{u} + (-\mathbf{w})$$



Example: Consider the rectangular prism below, where $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{BC} = \mathbf{b}$ and $\overrightarrow{AE} = \mathbf{c}$:



Express the following in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

(a) $\overrightarrow{EF} = \overrightarrow{AB} = \underset{\sim}{a}$

(b) $\overrightarrow{FB} = \overrightarrow{EA} = -\overrightarrow{AE} = -\underset{\sim}{c}$

(c) $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \underset{\sim}{a} + \underset{\sim}{b}$

(d) $\overrightarrow{AG} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CG} = \underset{\sim}{a} + \underset{\sim}{b} + \overrightarrow{AE}$
$= \underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$

(e) $\overrightarrow{DB} = \overrightarrow{DA} + \overrightarrow{AB} = \overrightarrow{CB} + \underset{\sim}{a}$
$= -\overrightarrow{BC} + \underset{\sim}{a} = -\underset{\sim}{b} + \underset{\sim}{a}$

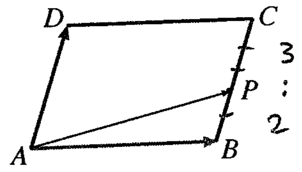
(f) $\overrightarrow{BH} = \overrightarrow{BA} + \overrightarrow{AD} + \overrightarrow{DH}$
$= -\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{EA}$
$= -\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$

1.2.5 Multiplying a vector by a scalar:

We can multiply any vector by a scalar (number). If the scalar is positive, then only the magnitude will be affected; that is, the direction will not change.

If the scalar is negative, the magnitude will change and the direction will be reversed.

Example: Let $ABCD$ be a parallelogram and let P divide the segment BC in the ratio 2 : 3. Express $\overrightarrow{AP}$ in terms of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$.



$$\overrightarrow{AP} = \overrightarrow{AB} + \overrightarrow{BP}$$

and $$\overrightarrow{BP} = \tfrac{2}{5}\overrightarrow{BC}$$

$$\Rightarrow \quad \overrightarrow{AP} = \overrightarrow{AB} + \tfrac{2}{5}\overrightarrow{BC}$$
$$= \overrightarrow{AB} + \tfrac{2}{5}\overrightarrow{AD}$$

1.2.6 Parallel vectors

Two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ are said to be parallel if and only if there is a non-zero scalar $k \in \mathbb{R}$ such that $\mathbf{u} = k\mathbf{v}$.

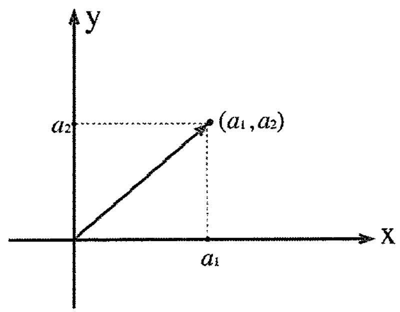
1.2.7 Properties of vectors

1. $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$ (vectors are commutative)
2. $(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$ (vectors are associative)
3. $\mathbf{a}+\mathbf{0}=\mathbf{a}$ (property of the zero vector)
4. $\mathbf{a}+(-\mathbf{a})=\mathbf{0}$ (property of the negative vector)
5. $m(\mathbf{a}+\mathbf{b})=m\mathbf{a}+m\mathbf{b}$ (scalar multiplication is distributive)

1.3 Rectangular coordinates

1.3.1 Rectangular coordinates in $\mathbb{R}^2$

To simplify problems involving vectors, we can introduce a rectangular coordinate system. Let $\mathbf{a}$ be a vector in the plane that has its starting point at the origin of a rectangular coordinate system. Its tip is at the point (a_1, a_2) as pictured below.



The coordinates (a_1, a_2) are called the **components** of $\mathbf{a}$, and we say that

$$\mathbf{a} = (a_1, a_2).$$

If $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ are both vectors with initial points at the origin, we say that they are *equivalent* exactly when their terminal points coincide. In terms of components, this means that

$$\mathbf{a} = \mathbf{b} \quad \text{if and only if} \quad a_1 = b_1 \text{ and } a_2 = b_2.$$

The algebraic operations that we have already defined for vectors are easy to perform when we write vectors in component form. We simply apply the operations *component-wise*.

Examples: Let $\mathbf{v} = (3, -2)$ and $\mathbf{w} = (1, 5)$. Calculate

(a) $\mathbf{v} + \mathbf{w} = (3,-2) + (1,5) = (4,3)$

(b) $\mathbf{v} - \mathbf{w} = (3,-2) - (1,5) = (2,-7)$

(c) $\mathbf{w} - \mathbf{v} = (1,5) - (3,-2) = (-2,7)$

(d) $3\mathbf{v} = 3(3,-2) = (9,-6)$

Homework: Let $\mathbf{v} = (1, -3)$ and $\mathbf{w} = (-1, -4)$. Repeat (a)-(d) above.

Answers: (a) $(0, -7)$ (b) $(2, 1)$ (c) $(-2, -1)$ (d) $(3, -9)$

1.3.2 Rectangular coordinates in $\mathbb{R}^n$

We can extend the definition of a vector in component form to any number of components we like. So $\mathbf{v} = (1, 4, 7)$ is a vector in $\mathbb{R}^3$ whilst $\mathbf{w} = (1, 0, 2, 4, 7)$ is a vector in $\mathbb{R}^5$. The same algebraic operations apply in any $\mathbb{R}^n$.

Be careful, however, because you can only perform operations between vectors of the same dimension (that is, between vectors with the same number of components).

So:

$$(1, 4, 7) + (2, 0, 1) = (3,4,8)$$

but

$$(1, 4, 7) + (2, 0) = \text{not defined}$$

The properties of vectors in section 1.2.7 can be easily proved using rectangular coordinates in $\mathbb{R}^n$.

Example: Prove that $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ for vectors $\mathbf{a}$, $\mathbf{b}$ in $\mathbb{R}^3$.

Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be vectors in $\mathbb{R}^3$. Then

$$\begin{aligned}
\mathbf{a} + \mathbf{b} &= (a_1, a_2, a_3) + (b_1, b_2, b_3) \\
&= (a_1 + b_1, a_2 + b_2, a_3 + b_3) && \text{(by definition of vector addition)} \\
&= (b_1 + a_1, b_2 + a_2, b_3 + a_3) && \text{(since real numbers are commutative)} \\
&= (b_1, b_2, b_3) + (a_1, a_2, a_3) && \text{(by definition of vector addition)} \\
&= \mathbf{b} + \mathbf{a}
\end{aligned}$$

1.3.3 Vectors with initial points not at the origin

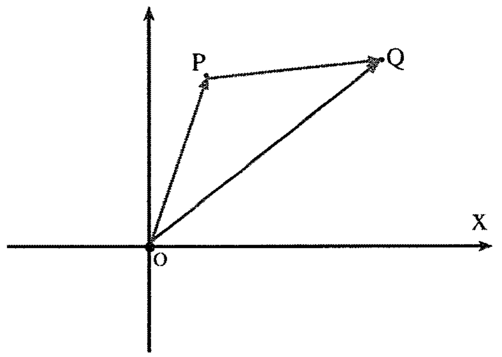
Sometimes a vector is positioned so that its initial point is not at the origin. Suppose that the vector $\overrightarrow{PQ}$ has initial point $P(x, y, z)$ and terminal point $Q(a, b, c)$.

We can work out the components of $\overrightarrow{PQ}$ as follows:

$$\begin{aligned}
\overrightarrow{PQ} &= \overrightarrow{PO} + \overrightarrow{OQ} \qquad = -\overrightarrow{OP} + \overrightarrow{OQ} \\
&= \overrightarrow{OQ} - \overrightarrow{OP} \\
&= (a, b, c) - (x, y, z) \\
&= (a - x, b - y, c - z).
\end{aligned}$$

So:

$$\boxed{\overrightarrow{PQ} = (a - x, b - y, c - z)}$$

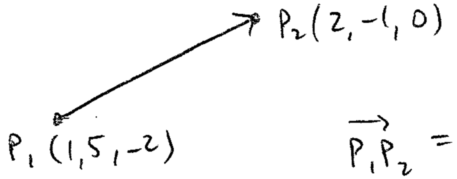


An easy way to remember this is

$$\overrightarrow{PQ} = \text{endpoint} - \text{initial point}$$
$$= Q - P$$

where the Q and P here may be thought of as representing the position vectors $\overrightarrow{OQ}$ and $\overrightarrow{OP}$.

Example: If P_1 has coordinates $(1,5,-2)$ and P_2 has coordinates $(2,-1,0)$, find $\overrightarrow{P_1P_2}$.



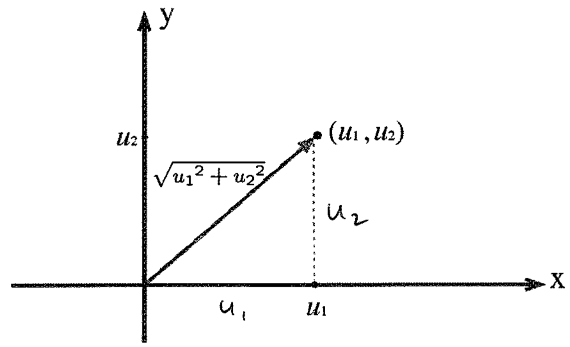
$$\overrightarrow{P_1P_2} = P_2 - P_1$$
$$= (2,-1,0) - (1,5,-2)$$
$$= (1,-6,2)$$

Homework: If A has coordinates $(2,-3,-7)$ and B has coordinates $(1,-3,2)$, find $\overrightarrow{AB}$.

Answer: $(-1,0,9)$

1.4 Length of a vector

As already discussed, every vector has a magnitude or *length*. We can find the length of a vector in component form by using Pythagoras' Theorem.



The **length** of $\mathbf{u}$ (also called the **norm** of $\mathbf{u}$), denoted as $\|\mathbf{u}\|$, is given by:

$$\|\mathbf{u}\| = \sqrt{{u_1}^2 + {u_2}^2}$$

We can extend this definition to vectors in $\mathbb{R}^3$ (or indeed $\mathbb{R}^n$ for any $n \in \mathbb{N}$) as follows.

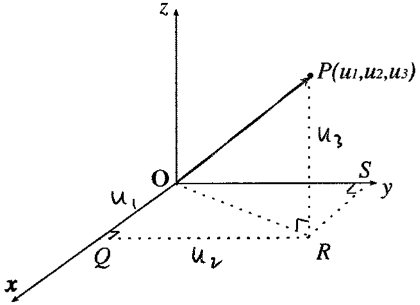
Firstly, for $\mathbf{u} \in \mathbb{R}^3$ the length of $\mathbf{u}$ is given by:

$$\|\mathbf{u}\| = \sqrt{{u_1}^2 + {u_2}^2 + {u_3}^2}$$

For $\mathbf{u} \in \mathbb{R}^n$, this generalises to:

$$\|\mathbf{u}\| = \sqrt{{u_1}^2 + {u_2}^2 + \cdots + {u_n}^2}$$

To see the $\mathbb{R}^3$ case, first note that the length of the line segment OR below is $\sqrt{u_1{}^2 + u_2{}^2}$.



Therefore

$$\begin{aligned} \left\|\overrightarrow{OP}\right\| &= \sqrt{\left\|\overrightarrow{OR}\right\|^2 + \left\|\overrightarrow{RP}\right\|^2} \\ &= \sqrt{\left(\sqrt{u_1{}^2 + u_2{}^2}\right)^2 + u_3{}^2} \\ &= \sqrt{u_1{}^2 + u_2{}^2 + u_3{}^2} \qquad \text{as claimed.} \end{aligned}$$

Example: Find the length of the vectors (a) $(1, 2)$ (b) $(-1, 3, -2)$.

(a) $\|(1,2)\| = \sqrt{1^2 + 2^2}$
$= \sqrt{5}$

(b) $\|(-1,3,-2)\| = \sqrt{(-1)^2 + 3^2 + (-2)^2}$
$= \sqrt{1 + 9 + 4}$
$= \sqrt{14}$

Homework: Find the length of the vectors (a) $(-2, -1)$ (b) $(-3, 6, -2)$.

Answers: (a) $\sqrt{5}$ (b) 7

1.4.1 Properties of the norm

For any vector $\mathbf{u}$, the following properties are always true:

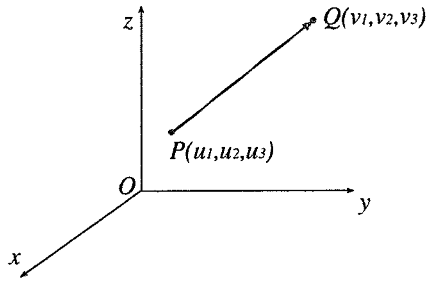
1. $\|\mathbf{u}\| \geq 0$ and $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$.
2. $\|k\mathbf{u}\| = |k|\|\mathbf{u}\|$ for any $k \in \mathbb{R}$.
3. $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

Example: We will prove property 2 when $\mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$.

$$\begin{aligned} \|k\mathbf{u}\| &= \|k\,(u_1, u_2, u_3)\,\| \\ &= \|\,(ku_1, ku_2, ku_3)\,\| \\ &= \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2} \\ &= \sqrt{k^2\left((u_1)^2 + (u_2)^2 + (u_3)^2\right)} \\ &= |k|\sqrt{(u_1)^2 + (u_2)^2 + (u_3)^2} \\ &= |k|\|\mathbf{u}\| \end{aligned}$$

1.4.2 Distance between two points

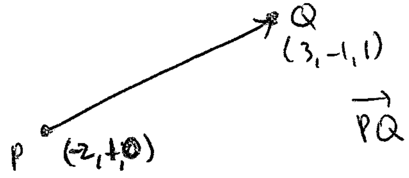
If $P\,(u_1, u_2, u_3)$ and $Q\,(v_1, v_2, v_3)$ are two points in $\mathbb{R}^3$, then the distance between them is just the length of the vector $\overrightarrow{PQ}$:



We know that $\overrightarrow{PQ} = (v_1 - u_1, v_2 - u_2, v_3 - u_3)$, so the distance between P and Q is

$$\begin{aligned} \text{dist}(P, Q) &= \left\|\overrightarrow{PQ}\right\| \\ &= \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2}. \end{aligned}$$

Example: Find the distance between the points $P(-2,1,0)$ and $Q(3,-1,1)$.



$$\overrightarrow{PQ} = (3,-1,1) - (-2,1,0)$$
$$= (5,-2,1)$$

$$\text{dist}(P,Q) = \| \overrightarrow{PQ} \|$$
$$= \| (5,-2,1) \|$$
$$= \sqrt{5^2 + (-2)^2 + 1^2} = \sqrt{25+4+1} = \sqrt{30}$$

Homework: Find the distance between the points $A(-3,1,-1)$ and $B(2,0,-1)$.

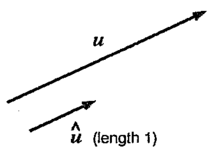
Answer: $\sqrt{26}$

1.4.3 Unit vectors

A vector with length 1 is called a **unit vector**. If $\mathbf{u}$ is any vector, we can construct a unit vector in the same direction as $\mathbf{u}$ by simply dividing the vector $\mathbf{u}$ by its length $\|\mathbf{u}\|$.

We use the notation $\hat{\mathbf{u}}$ to denote the unit vector in the direction of $\mathbf{u}$:

$$\boxed{\hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|}}$$



We can check that $\hat{\mathbf{u}}$ has length 1 as follows:

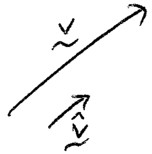
$$\|\hat{\mathbf{u}}\| = \left\| \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\| = \left\| \frac{1}{\|\mathbf{u}\|} \cdot \mathbf{u} \right\| = \frac{1}{\|\mathbf{u}\|} \|\mathbf{u}\| = 1$$

Example: Is the vector $\mathbf{v} = (1,-2,1)$ a unit vector? If it is not, find a unit vector in the same direction as $\mathbf{v}$.

$$\|\underset{\sim}{v}\| = \|(1,-2,1)\| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$$

$\Rightarrow$ no, not a unit vector.

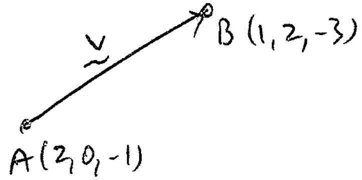
A unit vector in the same direction is



$$\hat{\underset{\sim}{v}} = \frac{\underset{\sim}{v}}{\|\underset{\sim}{v}\|} = \frac{(1,-2,1)}{\sqrt{6}} = \left(\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right)$$

Exercise: check this has length 1 by finding $\|\hat{\underset{\sim}{v}}\|$.

Example: Let $\mathbf{v}$ be a vector from $A(2,0,-1)$ to $B(1,2,-3)$. Find two unit vectors parallel to $\mathbf{v}$.

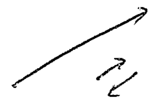


$$\underset{\sim}{v} = \overrightarrow{AB} = (1,2,-3) - (2,0,-1)$$
$$= (-1,2,-2)$$

and $\|\underset{\sim}{v}\| = \sqrt{(-1)^2 + 2^2 + (-2)^2} = \sqrt{9} = 3$

$$\Rightarrow \hat{\underset{\sim}{v}} = \frac{\underset{\sim}{v}}{\|\underset{\sim}{v}\|} = \frac{(-1,2,-2)}{3} = \left(-\frac{1}{3}, \frac{2}{3}, \frac{-2}{3} \right)$$

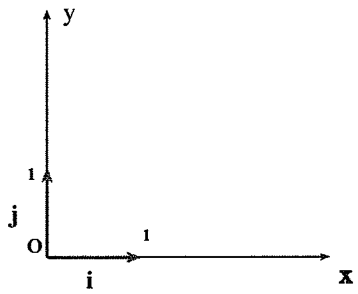
$\Rightarrow$ 2 unit vectors parallel to $\underset{\sim}{v}$ are



$$\pm \hat{\underset{\sim}{v}} = \pm \left(-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right)$$

1.5 Standard unit vectors

We now introduce some special unit vectors. Let $\mathbf{i} = (1,0)$ and $\mathbf{j} = (0,1)$. It is easy to see that both of these vectors have length 1. It is also easy to see that the vectors lie along the $x-$axis and $y-$axis respectively.



They are called the *standard unit vectors* in $\mathbb{R}^2$.

Every vector in $\mathbb{R}^2$ can be expressed in terms of $\mathbf{i}$ and $\mathbf{j}$.

Example: Express the vectors $\mathbf{v} = (1,2)$ and $\mathbf{u} = (-1,3)$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

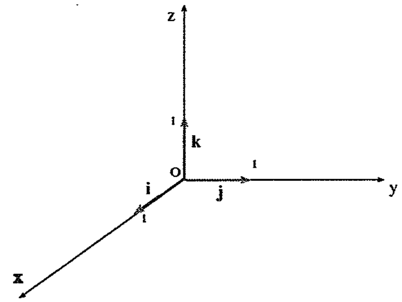
$$\underset{\sim}{v} = (1,2) = 1(1,0) + 2(0,1)$$
$$= 1\underset{\sim}{i} + 2\underset{\sim}{j}$$

$$\underset{\sim}{u} = (-1,3) = -1(1,0) + 3(0,1)$$
$$= -\underset{\sim}{i} + 3\underset{\sim}{j}$$

We can extend the definition of standard unit vectors to $\mathbb{R}^3$ by viewing $\mathbf{i}$ and $\mathbf{j}$ as vectors in $\mathbb{R}^3$ and introducing another unit vector in the z direction, which we call $\mathbf{k}$.

The standard unit vectors in $\mathbb{R}^3$ are therefore:

$$\mathbf{i} = (1,0,0),\ \mathbf{j} = (0,1,0) \text{ and } \mathbf{k} = (0,0,1).$$



Every vector in $\mathbb{R}^3$ can be expressed in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

Example: Express the vectors $\mathbf{v} = (-3,1,2)$ and $\mathbf{u} = (-1,1,-2)$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

$$\underset{\sim}{v} = (-3,1,2) = -3(1,0,0) + (0,1,0) + 2(0,0,1)$$
$$= -3\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$$

$$\underset{\sim}{u} = (-1,1,-2) = -\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$$

In general, if $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$, then

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$$

Similarly, if $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$, then

$$\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$$

These two notations (rectangular coordinates or $\mathbf{i}, \mathbf{j}, \mathbf{k}$) may be used interchangeably, and the same algebraic operations apply.

Warning! It is important to specify whether a vector of the form $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j}$ is in $\mathbb{R}^2$ or $\mathbb{R}^3$. If $\mathbf{u}$ is in $\mathbb{R}^3$ we can write $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j} + 0\mathbf{k}$ to be clearer.

Example: If $\mathbf{u} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, find

1. $\mathbf{u} + \mathbf{v} = (i + 3j - 2k) + (2i - j + 3k)$

   $= 3i + 2j + k$

2. $\mathbf{v} - \mathbf{u} = 2i - j + 3k - (i + 3j - 2k)$

   $= i - 4j + 5k$

3. $2\mathbf{u} + 3\mathbf{v} = 2(i + 3j - 2k) + 3(2i - j + 3k)$

   $= 2i + 6j - 4k + 6i - 3j + 9k$

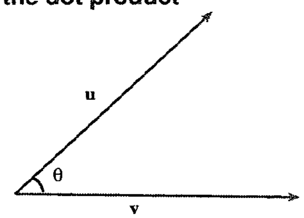
   $= 8i + 3j + 5k$

1.6 The dot product

We have seen how to multiply a vector by a scalar but we have not yet discussed the idea of multiplying a vector by another vector. In vector algebra there are two different ideas about multiplication of vectors. One is called the *dot product* which we discuss now, the other, the *cross product*, will be introduced in Calculus 2.

Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ and assume that the two vectors have been positioned so that their tails meet. We define the *angle between* $\mathbf{u}$ and $\mathbf{v}$ to be the angle θ such that $0 \le \theta \le \pi$ as pictured below.

1.6.1 Definition of the dot product



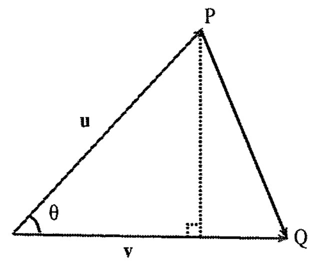
The **dot product** of two vectors $\mathbf{u}$ and $\mathbf{v}$, also known as the **scalar product**, is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined as:

$$\boxed{\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos(\theta)}$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

To calculate the dot product, we need to first calculate the length of $\mathbf{u}$, the length of $\mathbf{v}$ and the cosine of θ. Since all of these quantites are scalars, the dot product will be a scalar quantity (not a vector)!

Given u and v, we can easily find $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$, but we also need to know $\cos(\theta)$. This can be found using the Law of Cosines.



The Law of Cosines says that

$$\|\overrightarrow{PQ}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos(\theta).$$

Since $\overrightarrow{PQ} = -\mathbf{u} + \mathbf{v} = \mathbf{v} - \mathbf{u}$, this can be rearranged to give:

$$\cos(\theta) = \frac{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{v} - \mathbf{u}\|^2}{2\|\mathbf{u}\|\|\mathbf{v}\|}.$$

Example: Let $\mathbf{u} = (0, 0, 1)$ and $\mathbf{v} = (0, 2, 2)$. Find $\mathbf{u} \cdot \mathbf{v}$, by first finding $\cos(\theta)$ using the Law of Cosines.

$$\cos\theta = \frac{\|u\|^2 + \|v\|^2 - \|v-u\|^2}{2\|u\|\|v\|}$$

$$= \frac{1^2 + (\sqrt{8})^2 - (\sqrt{5})^2}{2 \cdot 1 \cdot \sqrt{8}}$$

$$= \frac{1+8-5}{2\sqrt{8}}$$

$$= \frac{4}{2\sqrt{8}} = \frac{2}{\sqrt{8}}$$

$\|u\| = \sqrt{0^2+0^2+1^2} = 1$

$\|v\| = \sqrt{0^2+2^2+2^2} = \sqrt{8}$

$v - u = (0,2,2) - (0,0,1)$
$= (0,2,1)$

$\|v-u\| = \sqrt{0^2+2^2+1^2}$
$= \sqrt{5}$

$$\Rightarrow u \cdot v = \|u\|\|v\|\cos\theta = 1 \cdot \cancel{\sqrt{8}} \cdot \frac{2}{\cancel{\sqrt{8}}} = 2$$

1.6.2 Important properties of the dot product

Let $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ and $\lambda \in \mathbb{R}$.

1. The dot product of two vectors is a scalar.
2. The order of multiplication is not important (that is, the dot product is *commutative*), so
$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}.$$
3. The dot product is *distributive*; that is,
$$\mathbf{v} \cdot (\mathbf{u} + \mathbf{w}) = \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{w}.$$
4. $\lambda(\mathbf{u} \cdot \mathbf{v}) = (\lambda\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (\lambda\mathbf{v})$.
5. If $\mathbf{u}$ and $\mathbf{v}$ are perpendicular, then $\mathbf{u} \cdot \mathbf{v} = 0$.
6. The dot product of a vector $\mathbf{u}$ with itself satisfies $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$.
7. $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

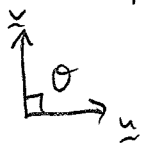
Some questions

1. Can you take the dot product of three vectors?

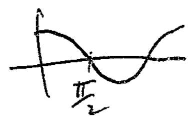
No. The dot product of 2 vectors is a scalar, so cannot take dot product of this with another vector.

2. Why is 5 true?

If u and v are perpendicular, then $\theta = \frac{\pi}{2}$.



$$\Rightarrow u \cdot v = \|u\|\|v\|\cos(\tfrac{\pi}{2})$$
$$= \|u\|\|v\| \cdot 0$$
$$= 0$$



3. Why is 6 true?

The angle between $\underset{\sim}{u}$ and itself is 0.

$$\Rightarrow \underset{\sim}{u}\cdot\underset{\sim}{u} = \|\underset{\sim}{u}\|\,\|\underset{\sim}{u}\| \cos 0 = \|\underset{\sim}{u}\|\,\|\underset{\sim}{u}\|\cdot 1 = \|\underset{\sim}{u}\|^2$$

4. What if u and v are parallel?

$\theta = 0$: If $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are in same direction:

$$\underset{\sim}{u}\cdot\underset{\sim}{v} = \|\underset{\sim}{u}\|\,\|\underset{\sim}{v}\| \cos 0 = \|\underset{\sim}{u}\|\,\|\underset{\sim}{v}\|$$

$\theta = \pi$: If $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are in opposite direction:

$$\underset{\sim}{u}\cdot\underset{\sim}{v} = \|\underset{\sim}{u}\|\,\|\underset{\sim}{v}\| \cos \pi = \|\underset{\sim}{u}\|\,\|\underset{\sim}{v}\|\cdot(-1) = -\|\underset{\sim}{u}\|\,\|\underset{\sim}{v}\|$$

1.6.3 An easier way to calculate the dot product

It would be useful for computational purposes, to have a way of calculating the dot product from the components of the vectors rather than from the angle between them.

Remember that our definition of the dot product is:

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos(\theta)$$

and that the law of cosines tells us:

$$\|\mathbf{v}-\mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos(\theta)$$

which rearranged gives:

$$\cos(\theta) = \frac{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{v}-\mathbf{u}\|^2}{2\|\mathbf{u}\|\|\mathbf{v}\|}.$$

cos(θ)

We can combine these two equations to obtain, for $\mathbf{u},\mathbf{v}\in\mathbb{R}^3$:

$$\begin{aligned}
\mathbf{u}\cdot\mathbf{v} &= \cancel{\|\mathbf{u}\|\|\mathbf{v}\|}\left(\frac{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{v}-\mathbf{u}\|^2}{2\cancel{\|\mathbf{u}\|\|\mathbf{v}\|}}\right)\\
&= \frac{1}{2}\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{v}-\mathbf{u}\|^2\right)\\
&= \frac{1}{2}\Big(({u_1}^2 + {u_2}^2 + {u_3}^2) + ({v_1}^2 + {v_2}^2 + {v_3}^2)\\
&\qquad - \big((v_1-u_1)^2 + (v_2-u_2)^2 + (v_3-u_3)^2\big)\Big)\\
&= \frac{1}{2}(2u_1v_1 + 2u_2v_2 + 2u_3v_3)\\
&= u_1v_1 + u_2v_2 + u_3v_3
\end{aligned}$$

expand and cancel terms

So for any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$

$$\boxed{\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.}$$

An analogous result is true in $\mathbb{R}^n$ for any $n\in\mathbb{N}$.
This gives us a much easier way of computing the dot product!

Example: Let $\mathbf{u} = (0,0,1)$ and $\mathbf{v} = (0,2,2)$, as in the example on p42. Find $\mathbf{u}\cdot\mathbf{v}$.

$$\begin{aligned}
\underset{\sim}{u}\cdot\underset{\sim}{v} &= (0,0,1)\cdot(0,2,2)\\
&= 0\times 0 + 0\times 2 + 1\times 2\\
&= 0 + 0 + 2 = 2
\end{aligned}$$

much easier than p42!

Example: Let $\mathbf{u} = (2,3,-1)$ and $\mathbf{v} = (4,5,0)$. Calculate $\mathbf{u}\cdot\mathbf{v}$.

$$\begin{aligned}
\underset{\sim}{u}\cdot\underset{\sim}{v} &= (2,3,-1)\cdot(4,5,0)\\
&= 2\cdot 4 + 3\cdot 5 + (-1)\cdot 0\\
&= 8 + 15 + 0 = 23
\end{aligned}$$

Homework: Calculate $\mathbf{u}\cdot\mathbf{v}$ if $\mathbf{u} = (3,2,1)$ and $\mathbf{v} = (1,1,3)$.
Answer: 8

$\in \mathbb{R}^2$

Example: Let $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$. Find $\mathbf{u} \cdot \mathbf{v}$.

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = 3 \cdot 2 + 2 \cdot (-1) = 6 - 2 = 4$$

Example: Let $\mathbf{u} = 2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$. Find $\mathbf{u} \cdot \mathbf{v}$.

$$\begin{aligned}\underset{\sim}{u} \cdot \underset{\sim}{v} &= 2 \cdot 2 + 2 \cdot (-1) + 3 \cdot (-2) \\ &= 4 - 2 - 6 \\ &= -4\end{aligned}$$

Homework: If $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{w} = \mathbf{j} + \mathbf{k}$, calculate $\mathbf{u} \cdot \mathbf{v}$, $\mathbf{v} \cdot \mathbf{w}$, and $\mathbf{w} \cdot \mathbf{u}$.

Answers: 6 , 0 , −1

1.6.4 A couple of proofs

We can use this easy way of computing the dot product to prove some general properties of the dot product. For example:

1. Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be two vectors in $\mathbb{R}^3$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned}\lambda(\mathbf{u} \cdot \mathbf{v}) &= \lambda\,(u_1v_1 + u_2v_2 + u_3v_3) \\ &= \lambda u_1v_1 + \lambda u_2v_2 + \lambda u_3v_3 \\ &= (\lambda u_1)v_1 + (\lambda u_2)v_2 + (\lambda u_3)v_3 \\ &= (\lambda \mathbf{u}) \cdot \mathbf{v}.\end{aligned}$$

2. If $\mathbf{0}$ is the zero vector, we have:

$$\begin{aligned}\mathbf{u} \cdot \mathbf{0} &= (u_1, u_2, u_3) \cdot (0, 0, 0) \\ &= u_1 \times 0 + u_2 \times 0 + u_3 \times 0 \\ &= 0\end{aligned}$$

1.6.5 Finding the angle between vectors

Now that we know two different ways to calculate the dot product, we can combine the two to find the angle between two vectors.

We have

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= u_1v_1 + u_2v_2 + u_3v_3 \\ &= \|\mathbf{u}\|\|\mathbf{v}\| \cos(\theta).\end{aligned}$$

Rearranging the equation gives

$$\boxed{\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{u_1v_1 + u_2v_2 + u_3v_3}{\|\mathbf{u}\|\|\mathbf{v}\|}}$$

Recall that the angle between two vectors is always taken to be in the interval $[0, \pi]$.

Example: Find the angle θ between the vectors $\mathbf{u} = (1, -2, 0)$ and $\mathbf{v} = (3, 1, -2)$.

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \|\underset{\sim}{u}\| \|\underset{\sim}{v}\| \cos\theta$$

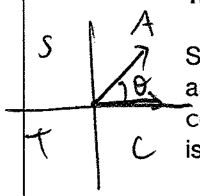
$$\begin{aligned}\Rightarrow \cos\theta &= \frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{\|\underset{\sim}{u}\| \|\underset{\sim}{v}\|} \\ &= \frac{1 \cdot 3 + (-2) \cdot 1 + 0 \cdot (-2)}{\sqrt{1^2 + (-2)^2 + 0^2} * \sqrt{3^2 + 1^2 + (-2)^2}} \\ &= \frac{3 - 2 + 0}{\sqrt{5}\sqrt{14}} = \frac{1}{\sqrt{5}\sqrt{14}}\end{aligned}$$

$$\Rightarrow \theta = \arccos\left(\frac{1}{\sqrt{5}\sqrt{14}}\right)$$

Homework: Find the angle between $\mathbf{u} = (-3, 0, 1)$ and $\mathbf{v} = (2, -2, 1)$.

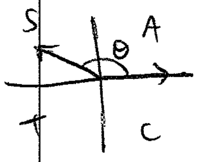
Answer: $\arccos\left(\frac{-5}{3\sqrt{10}}\right)$

1.6.6 Acute, obtuse, and right angles



cos +ve

Suppose we have two non-zero, non-parallel vectors $\mathbf{u}$ and $\mathbf{v}$, with angle θ between them. If θ is acute, ie. in the 1st quadrant, then $\cos(\theta)$ is positive. If θ is obtuse, ie. in the 2nd quadrant, then $\cos(\theta)$ is negative. If θ is a right angle, then $\cos(\theta) = 0$.



cos −ve

We also note that the sign of the dot product is the same as the sign of $\cos(\theta)$, since $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos(\theta)$, and $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$ are positive. This means that we can tell from the dot product whether the angle θ is acute or obtuse.

> If $\mathbf{u}$ and $\mathbf{v}$ are non-zero, non-parallel vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ and θ is the angle between them, then
>
> | | | |
> |---|---|---|
> | θ is acute | if and only if | $\mathbf{u} \cdot \mathbf{v} > 0$ |
> | θ is obtuse | if and only if | $\mathbf{u} \cdot \mathbf{v} < 0$ |
> | $\theta = \frac{\pi}{2}$ | if and only if | $\mathbf{u} \cdot \mathbf{v} = 0$ |

Example: Decide whether the angle between the following pairs of non-parallel vectors $\mathbf{u}$ and $\mathbf{v}$ is acute or obtuse:

(a) $\mathbf{u} = (1, -2, 0)$ and $\mathbf{v} = (3, 1, -2)$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = 1 \cdot 3 + (-2) \cdot 1 + 0(-2)$$
$$= 3 - 2 + 0 = 1 > 0 \Rightarrow \text{acute}$$

(b) $\mathbf{u} = (2, 1, 1)$ and $\mathbf{v} = (-3, -1, 2)$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = 2 \cdot (-3) + 1 \cdot (-1) + 1 \cdot 2$$
$$= -6 - 1 + 2 = -5 < 0 \Rightarrow \text{obtuse}$$

(c) $\mathbf{u} = (2, 0, 1)$ and $\mathbf{v} = (-1, -3, 2)$.

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = 2 \cdot (-1) + 0 \cdot (-3) + 1 \cdot 2$$
$$= -2 + 0 + 2 = 0 \Rightarrow \text{right angle / perpendicular}$$

1.6.7 Unit vectors and the dot product

Let's see what happens when we take dot products of pairs of vectors from the list $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

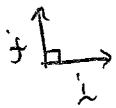
As we saw previously, if we take the dot product between any vector and itself, the answer will be the magnitude of the vector, squared. So

$$\mathbf{i} \cdot \mathbf{i} = 1, \ \ \mathbf{j} \cdot \mathbf{j} = 1, \ \text{ and } \mathbf{k} \cdot \mathbf{k} = 1$$

since they are all unit vectors. However,

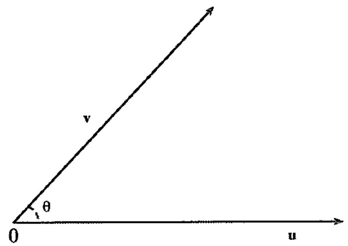
$$\mathbf{i} \cdot \mathbf{j} = 0, \ \ \mathbf{j} \cdot \mathbf{k} = 0, \ \text{ and } \mathbf{k} \cdot \mathbf{i} = 0$$

since they are pairwise perpendicular.



1.7 Scalar and vector projections

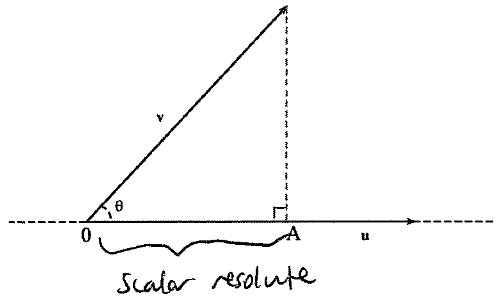
Consider the vectors $\mathbf{u}$ and $\mathbf{v}$ below, with angle θ between them. In this section, we look at how $\mathbf{v}$ can be viewed as the sum of one vector parallel to $\mathbf{u}$ and another vector perpendicular to $\mathbf{u}$.



1.7.1 The scalar resolute

First we obtain the component of $\mathbf{v}$ in the direction of $\mathbf{u}$ by the following construction.

Drop a perpendicular from the tip of $\mathbf{v}$ to the line through $\mathbf{u}$. This line is perpendicular to $\mathbf{u}$ and intersects the line through $\mathbf{u}$ at the point A.



Next we find the length of the line segment OA. From trigonometry we see that

$$\cos(\theta) = \frac{|OA|}{\|\mathbf{v}\|}$$

But from the definition of dot product, we have

$$\cos(\theta) = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

Combining these gives

$$\frac{|OA|}{\|\mathbf{v}\|} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

Then cancelling the $\|\mathbf{v}\|$s on each side gives

$$\begin{aligned} |OA| &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|} \\ &= \left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right)\cdot\mathbf{v} \end{aligned}$$

But we know that $\frac{\mathbf{u}}{\|\mathbf{u}\|} = \hat{\mathbf{u}}$, the unit vector in the direction of $\mathbf{u}$, so

$$|OA| = \hat{\mathbf{u}}\cdot\mathbf{v}$$

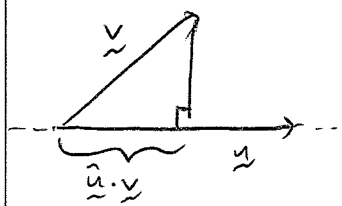
The length of this line segment OA is called the **scalar resolute** of $\mathbf{v}$ on $\mathbf{u}$ and is given by

$$\boxed{\hat{\mathbf{u}}\cdot\mathbf{v}}$$

Note: 'The vector we're projecting *onto* is the one with the hat (the unit vector).'

Example: Let $\mathbf{u} = (2, -1, 3)$ and $\mathbf{v} = (4, -1, 2)$. Find the scalar resolute of $\mathbf{v}$ on $\mathbf{u}$ and indicate this in a sketch.

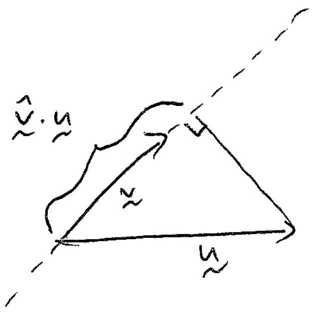
$$\begin{aligned} \text{scalar resolute} &= \hat{\underset{\sim}{u}}\cdot\underset{\sim}{v} \\ &= \frac{\underset{\sim}{u}}{\|\underset{\sim}{u}\|}\cdot\underset{\sim}{v} \\ &= \frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{\|\underset{\sim}{u}\|} \\ &= \frac{2\cdot 4 + (-1)\cdot(-1) + 3\cdot 2}{\sqrt{2^2+(-1)^2+3^2}} \\ &= \frac{13}{\sqrt{14}} \end{aligned}$$



We can also calculate the scalar resolute of $\mathbf{u}$ on $\mathbf{v}$. Note that the two quantities are not equal.

Example: For the same vectors, now find the scalar resolute of $\mathbf{u}$ on $\mathbf{v}$. Illustrate this in a sketch.

$$\hat{\underset{\sim}{v}} \cdot \underset{\sim}{u} = \frac{\underset{\sim}{v} \cdot \underset{\sim}{u}}{\|\underset{\sim}{v}\|} = \frac{13}{\sqrt{4^2+(-1)^2+2^2}} = \frac{13}{\sqrt{21}}$$

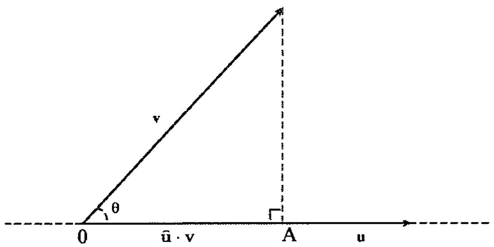


Homework: Let $\mathbf{u} = (1, 6, -2)$ and $\mathbf{v} = (-2, 0, -3)$. Find the scalar resolute of $\mathbf{v}$ on $\mathbf{u}$ and the scalar resolute of $\mathbf{u}$ on $\mathbf{v}$.

Answers: $\frac{4}{\sqrt{41}}$, $\frac{4}{\sqrt{13}}$

1.7.2 Vector resolutes

Let's go back to our diagram

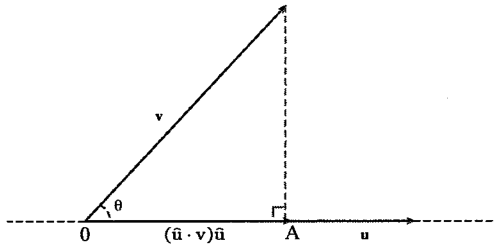


We know that the length of the line segment OA is just the scalar resolute $\hat{\mathbf{u}} \cdot \mathbf{v}$. If we think of the vector $\overrightarrow{OA}$, its direction is the same as the direction of $\mathbf{u}$. To construct a vector in the direction of $\mathbf{u}$ with the same length as OA we simply multiply $|OA|$ by a unit vector in the direction of $\mathbf{u}$.

This gives:

$$\boxed{\text{proj}_{\mathbf{u}}\mathbf{v} = (\hat{\mathbf{u}} \cdot \mathbf{v})\hat{\mathbf{u}}}$$

We call this vector the **vector resolute of $\mathbf{v}$ parallel to $\mathbf{u}$** or the **projection of $\mathbf{v}$ on $\mathbf{u}$**. We denote it by either $\text{proj}_{\mathbf{u}}\mathbf{v}$ or $\mathbf{v}_{||}$.

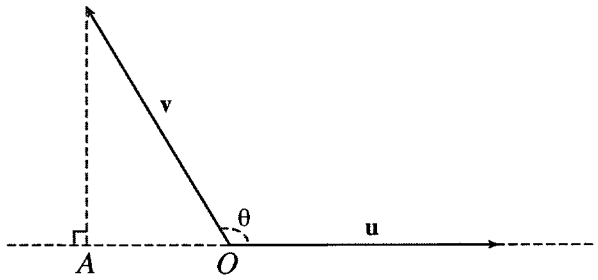


Example: Let $\mathbf{u} = (3, 1, -2)$ and $\mathbf{v} = (1, 0, 5)$. Find the scalar resolute of $\mathbf{v}$ on $\mathbf{u}$ and the projection of $\mathbf{v}$ on $\mathbf{u}$.

$$\text{scalar resolute} = \hat{\underset{\sim}{u}} \cdot \underset{\sim}{v} = \frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{\|\underset{\sim}{u}\|} = \frac{3 \cdot 1 + 1 \cdot 0 + (-2) \cdot 5}{\sqrt{3^2+1^2+(-2)^2}} = \frac{-7}{\sqrt{14}}$$

$$\text{proj}_{\underset{\sim}{u}} \underset{\sim}{v} = (\hat{\underset{\sim}{u}} \cdot \underset{\sim}{v})\hat{\underset{\sim}{u}} = -\frac{7}{\sqrt{14}} \cdot \frac{(3,1,-2)}{\sqrt{14}} = -\frac{7}{14}(3,1,-2) = -\frac{1}{2}(3,1,-2)$$

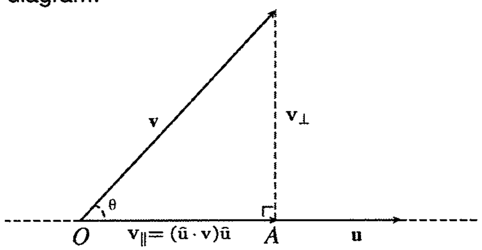
Note that we have obtained a negative scalar resolute. This simply means that the vector $\overrightarrow{OA}$ is in the opposite direction to $\mathbf{u}$, which occurs when the angle between $\mathbf{u}$ and $\mathbf{v}$ is obtuse.



Homework: Let $\mathbf{u} = (1,1,0)$ and $\mathbf{v} = (1,-2,3)$. Find the scalar resolute of $\mathbf{v}$ on $\mathbf{u}$ and the projection of $\mathbf{v}$ on $\mathbf{u}$.

Answers: $-\frac{1}{\sqrt{2}}$, $(-\frac{1}{2}, -\frac{1}{2}, 0)$

Back to the diagram:



It is easy to see that the vector $\mathbf{v}$ can be written as the sum

$$\mathbf{v} = \mathbf{v}_{\|} + \mathbf{v}_{\perp}$$

We can therefore calculate $\mathbf{v}_{\perp}$ as

$$\boxed{\mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{\|}}$$

This vector $\mathbf{v}_{\perp}$ is called the **vector resolute of $\mathbf{v}$ perpendicular to $\mathbf{u}$**, or the **orthogonal projection of $\mathbf{v}$ on $\mathbf{u}$**.

Example: Let $\mathbf{u} = (3,1,-2)$ and $\mathbf{v} = (1,0,5)$, as in the previous example. Find the orthogonal projection of $\mathbf{v}$ on $\mathbf{u}$.

From p64,

$$\underset{\sim}{v}_{\|} = \text{proj}_{\underset{\sim}{u}} \underset{\sim}{v} = -\tfrac{1}{2}(3,1,-2)$$

$$\Rightarrow \quad \underset{\sim}{v}_{\perp} = \underset{\sim}{v} - \underset{\sim}{v}_{\|}$$
$$= (1,0,5) - \left(-\tfrac{1}{2}(3,1,-2)\right)$$
$$= (1,0,5) + \left(\tfrac{3}{2}, \tfrac{1}{2}, -1\right)$$
$$= \left(\tfrac{5}{2}, \tfrac{1}{2}, 4\right)$$

Homework: Let $\mathbf{u} = (1,1,0)$ and $\mathbf{v} = (1,-2,3)$, as in the previous homework question. Find the orthogonal projection of $\mathbf{v}$ on $\mathbf{u}$.

Answer: $(\frac{3}{2}, -\frac{3}{2}, 3)$

Example: Let $\mathbf{u} = (-4,0,2)$ and $\mathbf{v} = (1,2,-7)$. Find the following, and illustrate with a sketch:

(i) the scalar resolute of $\mathbf{u}$ on $\mathbf{v}$

(ii) the projection of $\mathbf{u}$ on $\mathbf{v}$

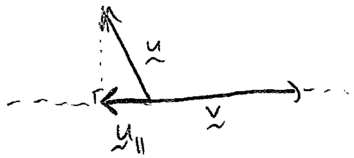
(iii) the orthogonal projection of $\mathbf{u}$ on $\mathbf{v}$

(i) $$\hat{\underset{\sim}{v}} \cdot \underset{\sim}{u} = \frac{\underset{\sim}{v} \cdot \underset{\sim}{u}}{\|\underset{\sim}{v}\|} = \frac{(-4)\cdot 1 + 0 + 2\cdot(-7)}{\sqrt{1^2 + 2^2 + (-7)^2}}$$
$$= \frac{-18}{\sqrt{54}}$$

(ii) $$\text{proj}_{\underset{\sim}{v}} \underset{\sim}{u} = \underset{\sim}{u}_{\|} = (\hat{\underset{\sim}{v}} \cdot \underset{\sim}{u})\, \hat{\underset{\sim}{v}}$$
$$= \frac{-18}{\sqrt{54}} \; \frac{(1,2,-7)}{\sqrt{54}}$$

$$= \frac{-18}{54}(1,2,-7)$$

$$= -\frac{1}{3}(1,2,-7)$$

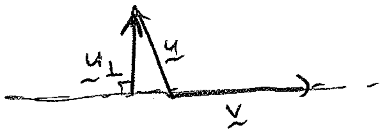


(iii) $u_\perp = u - u_\parallel$

$$= (-4,0,2) - \left(-\tfrac{1}{3}(1,2,-7)\right)$$

$$= (-4,0,2) + \tfrac{1}{3}(1,2,-7)$$

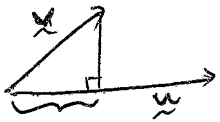
$$= \left(-\tfrac{11}{3}, \tfrac{2}{3}, -\tfrac{1}{3}\right)$$



1.7.3 Summary

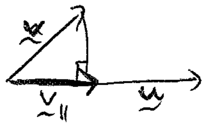
1. The **scalar resolute** of $\mathbf{v}$ on $\mathbf{u}$ is:

$$\hat{\mathbf{u}} \cdot \mathbf{v}$$



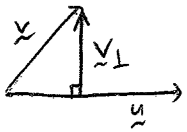
2. The **vector resolute of** $\mathbf{v}$ **parallel to** $\mathbf{u}$ or **projection of** $\mathbf{v}$ **on** $\mathbf{u}$ is:

$$\mathbf{v}_{\|} = (\hat{\mathbf{u}} \cdot \mathbf{v})\hat{\mathbf{u}}$$



3. The **vector resolute of** $\mathbf{v}$ **perpendicular to** $\mathbf{u}$ or **orthogonal projection of** $\mathbf{v}$ **on** $\mathbf{u}$ is:

$$\mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{\|}$$
$$= \mathbf{v} - (\hat{\mathbf{u}} \cdot \mathbf{v})\hat{\mathbf{u}}$$



1.8 Parametric curves

1.8.1 Introduction to parametric curves

Up to this point we have seen vectors of the form $\mathbf{u} = x\mathbf{i} + y\mathbf{j}$ where the components of the vectors have been constant. Suppose now that these components are functions of time; that is,

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$

where $t \in \mathbb{R}$.

Here $\mathbf{r}$ is a function from $\mathbb{R}$ to $\mathbb{R}^2$ since for each value of $t \in \mathbb{R}$, $\mathbf{r}(t)$ gives a vector in $\mathbb{R}^2$. This is thus called a **vector valued** function of a real variable. The functions $x(t)$ and $y(t)$ are called **parametric equations**, since they depend on the parameter t, and the resulting curve that $\mathbf{r}(t)$ traces out in $\mathbb{R}^2$ is called a **parametric curve**.

Such vector valued functions are particularly useful in applications, for example, describing the motion of a particle at any time t.

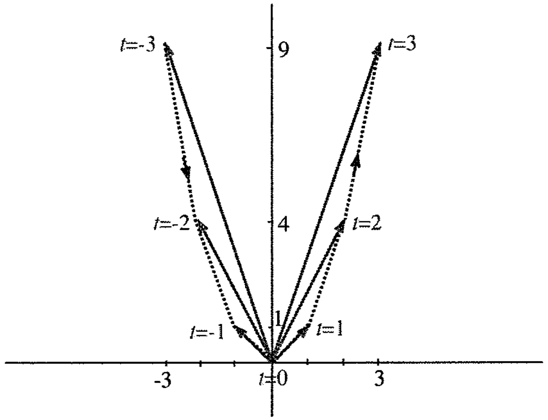
Let's look at a simple example. Let

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, \qquad \text{for } t \in \mathbb{R}.$$

This means that $x(t) = t$ and $y(t) = t^2$. To sketch the graph of $\mathbf{r}(t)$, we need to know the values of x and y for different values of t. A simple way to do this is to construct a table to give us a rough picture of what the curve looks like.

t	$x(t) = t$	$y(t) = t^2$	$\mathbf{r}$
-3	-3	9	$-3\mathbf{i} + 9\mathbf{j}$
-2	-2	4	$-2\mathbf{i} + 4\mathbf{j}$
-1	-1	1	$-\mathbf{i} + \mathbf{j}$
0	0	0	$\mathbf{0}$
1	1	1	$\mathbf{i} + \mathbf{j}$
2	2	4	$2\mathbf{i} + 4\mathbf{j}$
3	3	9	$3\mathbf{i} + 9\mathbf{j}$
4	4	16	$4\mathbf{i} + 16\mathbf{j}$

So for each ~~point~~ value of t we get a different vector, $\mathbf{r}(t)$.



We can see the *path* of the function by following the curve traced out by the heads of the vectors (indicated by the dotted line). Our job is to find the equation of this path.

1.8.2 Finding the equation of a path

To find the equation of a path defined by parametric equations, we need to solve the equations simultaneously. The aim is to eliminate the parameter t, to obtain a relationship between x and y.

Consider our example $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$. The coordinate in the x direction is t and the coordinate in the y direction is t^2. We put

$$x = t \quad \text{and} \quad y = t^2.$$

To eliminate the parameter we can simply substitute the first equation, which says $t = x$, into the second. Thus:

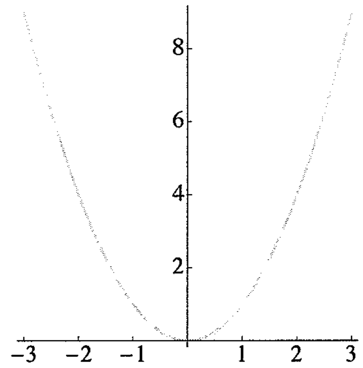
$$\begin{aligned} y &= t^2 \\ &= x^2 \end{aligned}$$

$$\text{i.e.} \quad y = x^2$$

So the equation of the path defined by the parametric equations is

$$y = x^2$$

Since $t \in \mathbb{R}$ and $x = t$, the domain of this function is $x \in \mathbb{R}$.



Example: Find the equation of the path of a particle whose position is given by

$$\mathbf{r}(t) = (t^2 - t)\mathbf{i} + 3t\,\mathbf{j}\,, \qquad \text{for } t \geq 0.$$

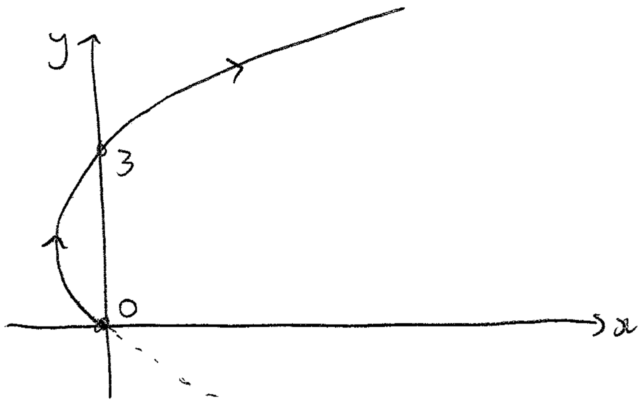
Sketch the graph of the path, indicating the direction of increasing t.

$x = t^2 - t \qquad y = 3t$

Idea: Take simpler equation, solve for t, and sub. into harder equation.

$y = 3t \Rightarrow t = \frac{y}{3}$

$$\begin{aligned} \Rightarrow x &= \left(\tfrac{y}{3}\right)^2 - \left(\tfrac{y}{3}\right) \\ &= \tfrac{y^2}{9} - \tfrac{y}{3} \quad \leftarrow \text{sideways parabola} \\ &= \tfrac{1}{9}(y^2 - 3y) = \tfrac{1}{9}y(y-3) \quad \leftarrow y\text{-intercepts } 0 \text{ and } 3 \end{aligned}$$



• since $t \geq 0$, $y = 3t \geq 0$

• as t increases, $y = 3t$ increases.

Homework: Find the equation of the path of a particle whose position is given by $\mathbf{r}(t) = t\mathbf{i} + (t^2+1)\mathbf{j}$, for $t \geq 0$.

Answer: $y = x^2 + 1$ for $x \geq 0$

Example: Find the equation of the path of a particle whose position is given by

$$\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}\,, \qquad \text{for } t \in \mathbb{R}.$$

Sketch the graph of the path.

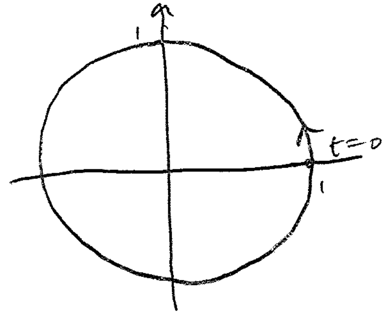
$x = \cos(t)$ $\qquad$ $y = \sin(t)$

$\cos^2 t + \sin^2 t = 1$

$\Rightarrow \quad x^2 + y^2 = 1$

circle! $\qquad$ centre $(0,0)$

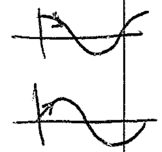
radius 1



• $t=0 \Rightarrow \underset{\sim}{r}(0) = \cos(0)\underset{\sim}{i} + \sin(0)\underset{\sim}{j} = 1\underset{\sim}{i} + 0\underset{\sim}{j} = (1,0)$

• as t increases from 0, $x = \cos t$ decreases

$y = \sin t$ increases



Homework: Find the equation of the path of a particle whose position is given by $\mathbf{r}(t) = \cos(t)\mathbf{i} + 2\sin(t)\mathbf{j}$, for $t \in \mathbb{R}$.

Answer: $x^2 + \frac{y^2}{4} = 1$

In the above example, the path described is a circle, which we could have simply sketched from the cartesian equation $x^2 + y^2 = 1$. However, the parametric equations give us more information about the motion of the particle:

- it tells us that at time $t = 0$ the particle is at the point $(1, 0)$.
- it tells us the direction of motion is anticlockwise.

  This can be seen in two ways:
  - As t increases from 0, $x = \cos(t)$ decreases, while $y = \sin(t)$ increases (consider the cos and sin graphs).
  - The point given by $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$ is simply at angle t around the unit circle, so as t increases, it moves in an anticlockwise direction.
- it tells us that it takes the particle time 2π to travel around the circle, i.e. its **period of motion** is 2π.
  - This is because $\mathbf{r}(t)$ first returns to its original position when $t = 2\pi$, since $\cos(t)$ and $\sin(t)$ both have period 2π.

Example: Find the equation of the path of a particle whose position is given by

$$\mathbf{r}(t) = \sin(2t)\mathbf{i} + \cos(2t)\mathbf{j}, \qquad \text{for } t \in \mathbb{R}.$$

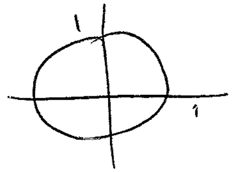
Sketch the graph of the path.

$x = \sin(2t)$ $\qquad$ $y = \cos(2t)$

$\sin^2(2t) + \cos^2(2t) = 1$

$x^2 + y^2 = 1$

Same circle!



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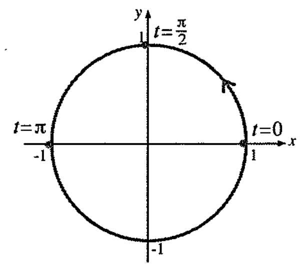
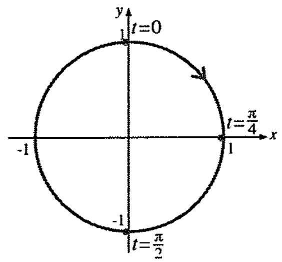
Here we have described the same path as the previous example, but for any given t we may not be at the same point on the path. So for example, when $t = 0$, in the first example we are at the point $(1, 0)$ whereas in the second example, we are at the point $(0, 1)$.



Graph of $\cos(t)\mathbf{i} + \sin(t)\mathbf{j}$



Graph of $\sin(2t)\mathbf{i} + \cos(2t)\mathbf{j}$

In the first example, the particle moves anticlockwise with a period of 2π, while in the second example, the particle moves clockwise with a period of π.

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Example: The motion of a particle is described by

$$\mathbf{r}(t) = (3 - 5\cos(2t))\mathbf{i} + (1 - 2\sin(2t))\mathbf{j}, \qquad \text{for } t \geq 0.$$

Determine:

(a) the cartesian equation of the path

(b) the position of the particle at time

(i) $t = 0$ $\quad$ (ii) $t = \frac{\pi}{4}$ $\quad$ (iii) $t = \frac{\pi}{2}$

(c) the time taken by the particle to return to its original position

(d) the direction of motion.

(a) $x = 3 - 5\cos(2t)$ $\qquad$ $y = 1 - 2\sin(2t)$

$x - 3 = -5\cos(2t)$ $\qquad$ $y - 1 = -2\sin(2t)$

$\frac{x-3}{-5} = \cos(2t)$ $\qquad$ $\frac{y-1}{-2} = \sin(2t)$

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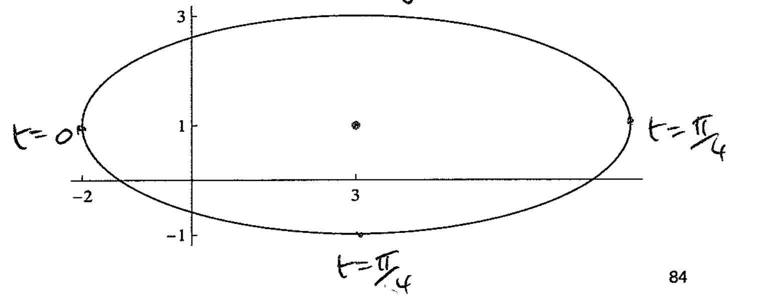
Know: $\cos^2(2t) + \sin^2(2t) = 1$

$\rightarrow \left(\frac{x-3}{-5}\right)^2 + \left(\frac{y-1}{-2}\right)^2 = 1$

$\frac{(x-3)^2}{5^2} + \frac{(y-1)^2}{2^2} = 1$

Ellipse:
- centre $(3, 1)$
- major axis length 10 parallel to x axis
- minor axis length 4



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(b) (i) $t=0$:

$$\underline{r}(0) = (3-5\cos 0)\underline{i} + (1-2\sin 0)\underline{j}$$
$$= (3-5(1))\underline{i} + (1-2(0))\underline{j}$$
$$= -2\underline{i} + \underline{j}$$

(ii) $t = \frac{\pi}{4}$:

$$\underline{r}\left(\tfrac{\pi}{4}\right) = \left(3-5\cos\left(2\left(\tfrac{\pi}{4}\right)\right)\right)\underline{i} + \left(1-2\sin\left(2\left(\tfrac{\pi}{4}\right)\right)\right)\underline{j}$$
$$= \left(3-5\cos\tfrac{\pi}{2}\right)\underline{i} + \left(1-2\sin\tfrac{\pi}{2}\right)\underline{j}$$
$$= (3-5\cdot 0)\underline{i} + (1-2\cdot 1)\underline{j}$$
$$= 3\underline{i} - \underline{j}$$

(iii) $t = \frac{\pi}{2}$:

$$\underline{r}\left(\tfrac{\pi}{2}\right) = \left(3-5\cos\left(2\left(\tfrac{\pi}{2}\right)\right)\right)\underline{i} + \left(1-2\sin\left(2\left(\tfrac{\pi}{2}\right)\right)\right)\underline{j}$$
$$= (3-5\cos\pi)\underline{i} + (1-2\sin\pi)\underline{j}$$
$$= (3-5(-1))\underline{i} + (1-2\cdot 0)\underline{j}$$
$$= 8\underline{i} + \underline{j}$$

(c) To complete motion around ellipse will take time $t=\pi$

(d) Direction of motion is anticlockwise.

Example: The motion of two particles is given by the equations

$\mathbf{r_1}(t) = (t+1)\mathbf{i} + (t^2-4t)\mathbf{j}$ and $\mathbf{r_2}(t) = (2t)\mathbf{i} + (6t-9)\mathbf{j}$

Determine:

(a) the point(s) at which the particles collide.

(b) the distance between the particles when $t=2$.

(a) Particles collide when $\underline{r}_1(t) = \underline{r}_2(t)$ for a certain t.

$$\Rightarrow (t+1)\underline{i} + (t^2-4t)\underline{j} = (2t)\underline{i} + (6t-9)\underline{j}$$

$\underline{i}$: $t+1 = 2t$
$1 = t \quad \Rightarrow t=1$

$\underline{j}$: $t^2-4t = 6t-9$

require both

$$t^2-10t+9=0$$
$$(t-1)(t-9)=0 \quad \Rightarrow t=1,\ t=9$$

$\Rightarrow t=1$.

i.e. the particles collide when $t=1$.

This is at:

$$\underline{r}_1(1) = (1+1)\underline{i} + (1^2-4\cdot 1)\underline{j}$$
$$= 2\underline{i} - 3\underline{j}$$

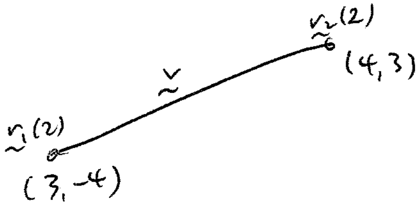
(Check: $\underline{r}_2(1) = (2\cdot 1)\underline{i} + (6-9)\underline{j} = 2\underline{i} - 3\underline{j}$) ✓

(b) when $t=2$:

$\underset{\sim}{r}_1(2) = (2+1)\underset{\sim}{i} + (2^2-8)\underset{\sim}{j} = 3\underset{\sim}{i} - 4\underset{\sim}{j} = (3,-4)$

$\underset{\sim}{r}_2(2) = 4\underset{\sim}{i} + (12-9)\underset{\sim}{j} = 4\underset{\sim}{i} + 3\underset{\sim}{j} = (4,3)$

$\Rightarrow$ distance between $\underset{\sim}{r}_1(2)$ and $\underset{\sim}{r}_2(2)$ is



$\underset{\sim}{v} = (4,3) - (3,-4)$
$= (1,7)$

$\Rightarrow$ dist $= \|\underset{\sim}{v}\|$
$= \|(1,7)\|$

$= \sqrt{1^2+7^2}$
$= \sqrt{50}$

Note: Since the particles collide at the point $(2,-3)$ (when $t=1$), the two paths must cross at this point. However, there may be other points where the paths cross, but where the particles do not collide since they are not there at the same time. (Imagine two people walking around a room. Their paths may cross, but they will only bump into eachother if they are at the same point *at the same time*.)

Homework: Find the equations of the two paths in this example and hence find all points where the paths cross.

Answer: $\mathbf{r}_1 : y = x^2 - 6x + 5$, $\mathbf{r}_2 : y = 3x - 9$.
Paths cross at $(2,-3)$ and $(7,12)$.