PROBLEM 7; TRANSFORMATIONS

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Introduction

In this poster, 'a mapping' should typically be taken to mean 'a mapping using some finite combination of the allowed transformations'. Similarly, 'to map' should typically be taken to mean 'to map using some finite combination of the allowed transformations'

Lemma 1

It is possible to map any rational number $\frac{a}{b}$ to $\frac{a}{b}+n$ for $n \in \mathbb{Z}$.
Proof by construction:
Let $T_3(x) = (T_2 \circ T_1^{-1})(x)$. Therefore,

$$T_3(x) = 1 - \frac{1}{\frac{1}{2-x}} = 1 - (2-x) = x - 1 \quad (1)$$

Similarly, let $T_4(x) = (T_1 \circ T_2^{-1})(x)$. Then,

$$T_4(x) = 2 - \frac{1}{\frac{1}{1-x}} = 2 - (1-x) = x + 1 \quad (2)$$

Clearly, repeated applications of either T_3 or T_4 allow one to add any integer to a rational number

Lemma 2

It is possible to map any rational number $\frac{a}{b}$ to a rational number $\frac{a^*}{b}$, where $0 \leq a^* < b$.

Proof:
Let $a = k * b + c$, where $k, c \in \mathbb{Z}$ and $0 \leq c < b$ (in other words, $c \equiv a \mod b$). Therefore,

$$\frac{a}{b} = \frac{k*b+c}{b} = k + \frac{c}{b} \quad (3)$$

By lemma 1, it is possible to map any rational number $\frac{a}{b}$ to $\frac{a}{b}+n$ for $n \in \mathbb{Z}$. Taking $n = -k$, $k + \frac{c}{b}$ can be mapped to $\frac{c}{b}$. Therefore, $\frac{a}{b}$ can be mapped to $\frac{c}{b}$. By the condition described above, $0 \leq c < b$, so a^* can be taken to equal c and therefore lemma 2 is true.

Lemma 3

It is possible to map any rational number $\frac{a}{b}$ to a rational number $\frac{b^*}{a}$.
Proof by construction:

$$T_2(\frac{a}{b}) = 1 - \frac{1}{\frac{a}{b}} = 1 - \frac{b}{a} = \frac{a-b}{a} \quad (4)$$

Taking $b^* = a - b$, lemma 3 is true.

An inductive proof that all rational numbers can be mapped to 0

If P is true for $n \Rightarrow P$ is true for $n+1$:
Consider a number $n \in \mathbb{N}$ for which the following statement (P) is true: for all $\phi \in \mathbb{N}$ where $0 < \phi < n$, there is a mapping of any rational number of the form $\frac{a}{\phi}$ to 0. We will show that this statement entails that a mapping exists for all rational numbers of the form $\frac{a}{n}$, which is equivalent to saying that P is true for $n+1$. By lemma 2, it is possible to map any rational number $\frac{a}{n}$ to a rational number $\frac{a^*}{n}$, where $0 \leq a^* < n$. Then, by lemma 3, one can map $\frac{a^*}{n}$ to $\frac{n^*}{a^*}$, as long as $a^* \neq 0$. Importantly, we need not consider $a^* = 0$ as if $a^* = 0$, $\frac{a^*}{n} = 0$, so a mapping to 0 has already been achieved. This means that $0 < a^* < n$, so it is possible to map $\frac{n^*}{a^*}$ to 0, by P. Therefore, if P is true for n, it is also true for $n+1$.

P is true for the base case:
The base case in this induction argument is $n = 1$, as for $n = 0$ the fraction $\frac{a}{n}$ is undefined. For $n = 1$, $\frac{a}{n} = a$. By lemma 1, all fractions of this form can be mapped to 0 by adding $-a$ to them. Therefore, the base case is true.

On the completeness of this induction argument:
This inductive process reaches all rational numbers as it proves that for all rationals with denominators within $\mathbb{N}$, there is a mapping that takes these numbers to 0. The numerator of these fractions is allowed to be any number within $\mathbb{Z}$, so this spans the entire rational field. Therefore, for all rational numbers $\frac{a}{b}$, there is a mapping that takes them to 0.

Solution

By following the proof that a mapping to 0 exists for all rational numbers, a method for finding such a mapping quickly emerges. Namely:

1. If $\frac{a}{b} = 0$, finished
2. If $\frac{a}{b} > 0$, apply T_3 until the numerator of the resulting fraction is between 0 and b
3. Else, apply T_4 until the same condition is met
4. Apply T_2
5. Repeat steps 1-4 until the denominator of the fraction is 1 (until the fraction has been mapped to an integer)
6. If the integer is greater than 0 then apply T_3 until the integer has been mapped to 0
7. Else, apply T_4 until the same condition is met

Other sets of transformations

Following the proof, it can be seen that for any set of transformations where lemmas 1, 2 and 3 are true, then one can use this set of transformations to map any rational number to 0. Furthermore, as lemma 1 entails lemma 2, as long as a set of transformations obeys these properties:

1. It is possible to map any rational number $\frac{a}{b}$ to $\frac{a}{b}+n$ for $n \in \mathbb{Z}$ (lemma 1)
2. It is possible to map any rational number $\frac{a}{b}$ to a rational number $\frac{b^*}{a}$ (lemma 3)

then this set of transformations can map any rational number to 0. However, this condition is merely sufficient, not necessary, so it is possible that there are sets of transformations on the rational numbers for which lemma 1 and lemma 3 are not true that can still map every rational number to 0.

Infinite mappings

Suppose:

$$T_1^n(x) = \frac{(n+1)x - n}{nx - (n-1)} \text{ for some } n \in \mathbb{N} \quad (5)$$

Then:

$$\begin{aligned} T_1^{n+1}(x) &= T_1(\frac{(n+1)x-n}{nx-(n-1)}) \\ &= 2 - \frac{1}{\frac{(n+1)x-n}{nx-(n-1)}} \\ &= \frac{2*((n+1)x-n) - (nx-(n-1))}{(n+1)x-n} \\ &= \frac{(n+2)x-(n+1)}{(n+1)x-n} \end{aligned} \quad (6)$$

The base case, $n = 1$, is clearly true as,

$$\begin{aligned} T_1(x) &= 2 - \frac{1}{x} \\ &= \frac{2x-1}{x} \end{aligned} \quad (7)$$

Therefore,

$$T_1^n(x) = \frac{(n+1)x-n}{nx-(n-1)} \text{ for all } n \in \mathbb{N} \quad (8)$$

And moreover,

$$T_1^n(x) = 1 + \frac{x-1}{n(x-1)+1} \quad (9)$$

From (9), it can be seen that,

$$\lim_{n\to\infty} T_1^n(x) = 1 \text{ for all } x \in \mathbb{R} \quad (10)$$

And as $T_2(1) = 0$,

$$T_2(\lim_{n\to\infty} T_1^n(x)) = 0 \text{ for all } x \in \mathbb{R} \quad (11)$$

Hence, for all real numbers, there is an mapping of infinite length to 0.

Complex numbers

Consider the result when one applies the supplied transformations to a general complex number $a + bi$,

$$T_1(a+bi) = \frac{2(a^2+b^2)-1}{a^2+b^2} + \frac{b}{a^2+b^2}i \quad (12)$$

$$T_2(a+bi) = \frac{a^2+b^2-1}{a^2+b^2} + \frac{b}{a^2+b^2}i \quad (13)$$

$$T_1^{-1}(a+bi) = \frac{2-a}{(2-a)^2+b^2} + \frac{b}{(2-a)^2+b^2}i \quad (14)$$

$$T_2^{-1}(a+bi) = \frac{1-a}{(1-a)^2+b^2} + \frac{b}{(1-a)^2+b^2}i \quad (15)$$

This shows that none of these transformations could map a complex number with non-zero imaginary component to a real number, and hence could never (in a finite number of steps) map all complex numbers to 0. However, the above argument 'infinite transformations' can be generalised to the complex numbers, as none of its steps are false for $x \in \mathbb{C}$. Consequently, any complex number can still be reduced to 0, but only using infinite transformations.