# Symmetric Functions

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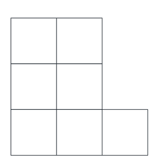
## Introduction

Symmetric functions are functions of multiple variables which remain unchanged under any permutation of the variables. Formally, if $X = \{x_i\}_{i=1}^{\infty}$ and $f$ is a formal power series in $X$, then $f$ is a symmetric function if for all $n \geq 1$ and $\pi \in S_n$ we have $\pi(f) = f$ (where $\pi$ is an element of a symmetric group which permutes the variables as $\pi(x_i) = x_{\pi(i)}$).
Symmetric functions arise in a number of settings in algebra, geometry, topology, combinatorics and more; but they are also fun mathematical objects in their own right.
There is an interesting zoo of classes of symmetric functions, all of which are indexed by partitions, so it will be useful to note the following.
**Definition:** A *partition* is a sequence

$$\lambda = (\lambda_1, \lambda_2, ..., \lambda_k, ...)$$

of non-negative integers such that $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k \geq ...$ with finitely many non-zero terms.
Partitions can be represented geometrically with a stack of boxes called a *Ferrers Diagram* where the bottom row has $\lambda_1$ boxes, the second bottom row has $\lambda_2$ boxes and so on. Below is the Ferrers diagram for the partition $\lambda = (3, 2, 2)$



## Schur Functions

Schur functions are a particular set of symmetric functions that arise in representation theory as they are the characters of the polynomial irreducible representations of the general linear groups.
To define Schur functions we introduce a special filling $T$ of a Ferrers diagram with positive integers and write $x^T$ to denote the monomial given by $x^T = \prod_{i \in T} x_i$, the filling is called a *Semistandard Tableaux* and the entries in the columns are strictly increasing from bottom to top and the entries in the rows are weakly increasing from left to right. We write $SST(\lambda)$ for the set of semistandard tableaux of shape $\lambda$ with entries from the positive integers. The *content* of a tableaux is the sequence $\{\mu_{i=1}^{\infty}\}$ where $\mu_i$ is the number of $i$'s in the tableaux.

**Definition:** For any partition $\lambda$

$$s_\lambda = \sum_{T \in SST(\lambda)} x^T$$

is the *Schur Function indexed by* $\lambda$.

**Example:** A few terms of the schur function $s_{(2,1)}$ and the corresponding Semistandard Tableaux:

$$s_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + 2x_1 x_2 x_3 + x_1^2 x_3 + x_1 x_3^2 + ...$$

$$\begin{array}{l}2\\1\;1\end{array} \quad \begin{array}{l}2\\1\;2\end{array} \quad \begin{array}{l}2\\1\;3\end{array} \quad \begin{array}{l}3\\1\;2\end{array} \quad \begin{array}{l}3\\1\;1\end{array} \quad \begin{array}{l}3\\1\;3\end{array}$$

**Theorem:** Schur functions are symmetric.
*Proof.* We need to show that for any $\lambda$ and all $\pi \in S_n$, $\pi(s_\lambda) = s_\lambda$. Notice that by construction each monomial in a Schur function has the same degree. Now suppose $\pi = (i, i+1)$ and choose a monomial $x_1^{\mu_1} ... x_n^{\mu_n}$. By definition the coefficient of this term in $s_\lambda$ is the number of semistandard tableaux of shape $\lambda$ and content $\mu_1, ..., \mu_n$. Similarly the coefficient of this term in $\pi(s_\lambda)$ is the coefficient of $x_1^{\mu_1} ... x_i^{\mu_{i+1}} x_{i+1}^{\mu_i} ... x_n^{\mu_n}$ in $s_\lambda$. By definition this is the number of semistandard tableaux with content $\mu_1, ..., \mu_{i-1}, \mu_{i+1}, \mu_i, ..., \mu_n$. Given a semistandard tableaux corresponding to this monomial, we say that $i$ is *paired* if there is an $i+1$ in it's column and *free* otherwise. The operation $\beta_i$ on semistandard tableaux $T$ which, row by row, replaces $a$ free $i$'s and $b$ free $i+1$'s with $b$ free $i$'s and $a$ free $i+1$'s results in a semistandard tableaux $\beta_i(T)$. Notice that the number of $i$'s in $\beta_i(T)$ is the number of paired $i+1$'s in $T$ plus the number of free $i+1$'s in $T$, this number is $\mu_i + 1$. Similarly the number of $i+1$'s in $\beta_i(T)$ is $\mu_i$. Hence $\beta_i(T)$ has content $\mu_1, ..., \mu_{i-1}, \mu_{i+1}, \mu_i, ..., \mu_n$. Next notice that the operation $\beta_i$ doesn't change which entries are free so if we applied $\beta_i$ to $\beta_i(T)$ we would change the tableaux back to the original $T$. This tells us that $\beta_i$ is a bijection between semistandard tableaux of content $\mu_1, ..., \mu_n$ and semistandard tableaux of content $\mu_1, ..., \mu_{i-1}, \mu_{i+1}, \mu_i, ..., \mu_n$. This bijection tells us that $x_1^{\mu_1} ... x_i^{\mu_{i+1}} x_{i+1}^{\mu_i} ... x_n^{\mu_n}$ in $\pi(s_\lambda)$ has the same coefficient as $x_1^{\mu_1} ... x_n^{\mu_n}$ in $s_\lambda$ . Hence $\pi(s_\lambda) = s_\lambda$. As any permutation can be written as a composition of simple transpositions the result follows for any permutation. $\square$

**Theorem:** For all $k \geq 0$, the set $\{s_\lambda \mid \sum_{i=1}^{\infty} \lambda_i = k\}$ is a basis for the set of all symmetric functions of homogenous degree $k$.

The above theorem presents an interesting problem: as any symmetric function of homogenous degree can be expressed as a linear combination of Schur functions. Consider for any two partitions $\mu$ and $\nu$ the product $s_\mu s_\nu$. As the product of two homogenous symmetric functions it is also a homogenous symmetric function and so by the above theorem can be expressed as a linear combination of Schur functions. The Littlewood-Richardson theorem tells us which Schur functions are in the linear combination and that the coefficients are positive integers.

## Littlewood-Richardson Rule

To state the rule we first need to build on what we know about tableaux. A *semistandard skew tableaux* $\lambda/\mu$ is the diagram of $\lambda$ with the $\mu$ boxes removed. For example a semistandard skew tableau with $\lambda = (3, 2, 1)$ and $\mu = (2, 1)$ is:

$$\begin{array}{lll}3 & & \\ \bullet & 2 & \\ \bullet & \bullet & 2\end{array}$$

The *reading word* of a tableaux is the word we get by reading the entries of each row left to right, starting with the top row and working down.

**Definition:** A word $a_1 a_2 ... a_n$ is a *Littlewood-Richardson Word* if every tail $a_k ... a_n$ of the word has at least as many copies of $j$ as it has copies of $j+1$ for every $j$.

**Theorem:** *(The Littlewood-Richardson Rule for Symmetric Functions)* For any partitions $\mu$ and $\nu$, let $c_{\mu,\nu}^{\lambda}$ be the number of semistandard skew tableaux of size $\lambda/\mu$ and content $\nu$ whose reading words are Littlewood-Richardson words. Then

$$s_\mu s_\nu = \sum_\lambda c_{\mu,\nu}^{\lambda} s_\lambda$$

**Example:** We calculate the product $s_{(2,1)} \cdot s_{(2,1)}$. As $(2, 1)$ is the content, we look for the semistandard skew tableaux with two $1$'s and one $2$. As the reading words need to be Littlewood-Richardson words, all words need to end in a $1$ because if it ends in a $2$ then that tail won't have any $1$'s. The possible semistandard skew tableaux are below:

$$\begin{array}{llll}\bullet & 2 & & \\ \bullet & \bullet & 1 & 1\end{array} \quad \begin{array}{llll}2 & & & \\ \bullet & & & \\ \bullet & \bullet & 1 & 1\end{array} \quad \begin{array}{lll}\bullet & 1 & 2 \\ \bullet & \bullet & 1\end{array} \quad \begin{array}{lll}2 & & \\ \bullet & 1 & \\ \bullet & \bullet & 1\end{array}$$

$$\begin{array}{lll}1 & & \\ \bullet & 2 & \\ \bullet & \bullet & 1\end{array} \quad \begin{array}{lll}2 & & \\ 1 & & \\ \bullet & & \\ \bullet & \bullet & 1\end{array} \quad \begin{array}{ll}1 & 2 \\ \bullet & 1 \\ \bullet & \bullet\end{array} \quad \begin{array}{ll}2 & \\ 1 & \\ \bullet & 1 \\ \bullet & \bullet\end{array}$$

And so the product can be expressed as:

$$s_{(2,1)} \cdot s_{(2,1)} = s_{(4,2)} + s_{(4,1,1)} + s_{(3,3)} + 2s_{(3,2,1)} + s_{(3,1,1,1)} + s_{(2,2,2)} + s_{(2,2,1,1)}$$

## Cauchy's Formula

It is possible to put an inner product on the space of symmetric functions and the Cauchy product formula is equivalent to the orthogonality of the Schur functions under *The Hall Inner Product*.
**Theorem:** *(Cauchy's Formula)* If $X = \{x_i\}_{i=1}^{\infty}$ and $Y = \{y_i\}_{i=1}^{\infty}$ are two sets of variables, then we have

$$\sum_\lambda s_\lambda(X) s_\lambda(Y) = \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \frac{1}{1 - x_i y_j}$$

where the sum on the left hand side is over all partitions.
This result is not only amazing for it's simplicity but also it's combinatorial interpretation. In proving this formula, one can consider the left hand side as a generating function for ordered pairs of semistandard tableaux $(P, Q)$ of the same shape, with weights $x^P y^Q$. The right hand side can be interpreted as a generating function (with respect to a construction not mentioned here) for $2 \times n$ arrays, $\pi = \begin{bmatrix} a_1 & a_2 & ... & a_n \\ b_1 & b_2 & ... & b_n \end{bmatrix}$ called *Generalized Permutations* for which $a_1 \leq a_2 \leq ... \leq a_n$ and if $a_1 = a_{i+1}$ then $b_i \leq b_{i+1}$.
A bijection $\Re$ from generalized partitions of length $n$ to pairs of semistandard tableaux with $n$ boxes and the same shape can be created via a correspondence of *Robinson, Schensted* and *Knuth*. Let $\Re(\pi) = (P(\pi), Q(\pi))$, where $P(\pi)$ is the semistandard tableaux constructed via the *RSK Insertion algorithm* with the entries in the bottom row of $\pi$. The RSK Insertion algorithm is a method of constructing semistandard tableaux by inserting entries into an empty tableaux one by one. If $c$ is the entry being added to the tableau, RSK states: starting with the bottom row, if $c$ is greater than or equal to every value in that row then add a box at the end of that row with $c$. If $c$ is not the largest entry, then say $b$ is the leftmost entry greater than $c$, replace $b$ with $c$ and repeat the process with $b$ in the next row up. Continue until a new box has been added to the tableaux. $Q(\pi)$ is the semistandard tableaux with entries from the top row of $\pi$. These are placed in the box that was added by inserting the corresponding bottom entry when making $P(\pi)$. The data of $Q(\pi)$ allows one to reconstruct $\pi$ given any pair of semistandard tableaux. Hence it can be shown that $\Re$ is a bijection.

**Example:** We calculate $\Re(\pi)$ for $\pi = \begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 & 2 \end{bmatrix}$. The steps of making $P(\pi)$:

$$\begin{array}{l}2\end{array} \rightarrow \begin{array}{ll}2 & 4\end{array} \rightarrow \begin{array}{ll}4 & \\ 2 & 3\end{array} \rightarrow \begin{array}{ll}4 & \\ 2 & \\ 1 & 3\end{array} \rightarrow \begin{array}{ll}4 & \\ 2 & 3 \\ 1 & 2\end{array}$$

And $Q(\pi)$ is:

$$\begin{array}{ll}3 & \\ 2 & 4 \\ 1 & 1\end{array}$$

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## References

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