Geometries of Homogeneous Spacetime Models

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Introduction

The defining tenet of General relativity asserts that the acceleration of objects is indistinguishable from their gravitation towards a distribution of mass. Such a viewpoint leads to the concept that an object's motion is representable by inertial paths, like those alluded to in Newton's First Law; called **geodesic null lines**, along a space defined by the presence of matter inside it.

Such a view is encapsulated in the **Einstein Field Equations**, an expression of local energy conservation:

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} + \Lambda g_{\alpha\beta} = T_{\alpha\beta} \quad (1)$$

This equates a description of distances and curvature (see LHS) to matter distribution (RHS).

Cosmological Interest

There are two main quantities of interest in the large-scale, cosmological viewpoint:

- A set, M, that defines the topology of the universe on the whole. We expect this, for intuitive reasons, to be (at least) locally isomorphic to $\mathbb{R}^{3+1}$.
- A metric, g, a 2-form that gives meaning to distance on M. That is, the line element ds is given by $ds^2 = g_{ab}dx^a dx^b$, for $\{dx^i\} \in T_p$M and $p \in$ M.

The pair (M, g) defines a **spacetime manifold**, which provides a description of a universe obeying **1**.

The extreme difficulty in solving **1** leads to an examination of metrics on particular topologies with suitable symmetries or physically intuitive properties. These do not, however, necessitate a faithful representation of our own universe, which we shall explore.

Einstein's Static Universe

An important example, and the first enumerated example of a cosmological model, is the 'Einstein Static Universe'.

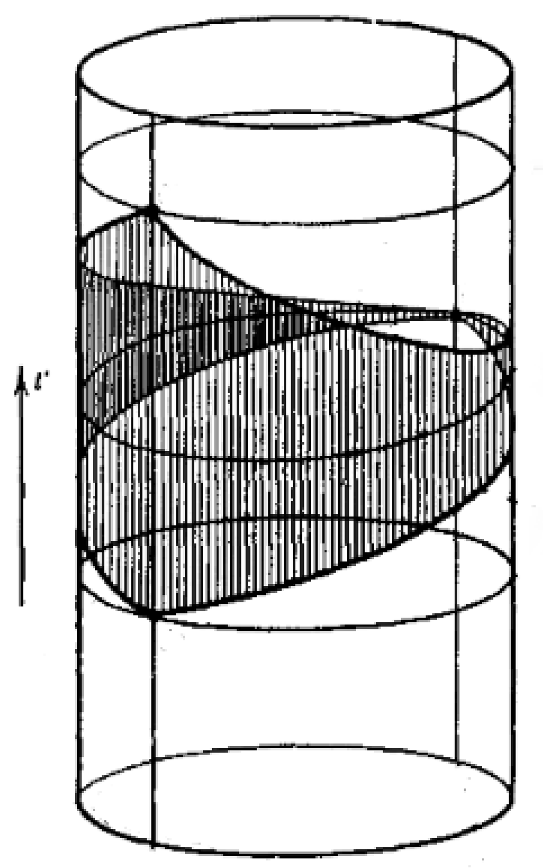

Figure 2: The Einstein Static universe with two dimensions suppressed; via [2]. Note the radius is constant; this is equiv. to constant $a(t)$. The shaded region is conformal to Minkowski space.

This looks for a static solution to **2** and **3**; i.e. s.t. $a(0) := a_0$ and $\dot{a}(0) = \ddot{a}(0) = 0$. This condition requires $\Lambda > 0$, $k = +1$; this universe ends up having a spatial component isomorphic to the 3-sphere, and hence is compact; i.e. space is finite in some sense.

Analysis of **2** and **3** leads to the parametric solution:

$$a(\lambda) = \frac{\alpha_1 \exp\{\alpha_2\sqrt{\Lambda/3}\lambda\}}{[\alpha_3 \exp\{\alpha_2\sqrt{\Lambda/3}\lambda\} + \alpha_4]^2 - \alpha_5} + \alpha_6$$

$$t(\lambda) = \alpha_6\lambda + \alpha_7 + 2\frac{3}{\Lambda}\log\frac{\alpha_2 + \alpha_8 + \alpha_3\exp\{\alpha_2\sqrt{\Lambda/3}\lambda\}}{\alpha_2 - \alpha_8 - \alpha_3\exp\{\alpha_2\sqrt{\Lambda/3}\lambda\}}$$

Where $\{\alpha_i\}$ are constants, in terms of the roots of a polynomial encountered during analysis, and have been omitted for brevity. See [4] for details.

By adjusting the initial conditions for the above about the static value a_0, one finds that any small perturbation results in a trajectory that does not lead back to a_0; a static solution is unstable. Because we do not expect isotropy and homogeneity to hold locally, this indicates that this universe is not physically significant.

Aspects of Robertson-Walker Spacetimes

Robertson-Walker spacetimes are characterised by their invariance under SO(3) action and two physical conditions:

- Homogeneity - a metric at any point on a spacelike hypersurface, i.e. the spatial aspect of the manifold at any given time, is isometric to that at any other point. Essentially, space appears the same at any point.
- Isotropy - there exists such an isometry that rotates a spatial vector into any other; making space the same from any angle. This condition is very strict.

It should be noted that our universe fulfils, to large-scale approximation, these conditions.[1] However, at periods in the universe's history this may not remain true - a true-to-reality model may not be a Robertson-Walker space.

Under the symmetry condition, $\mathrm{M} \cong \mathbb{R} \times \mathbb{S}^3$, and the other two lead to three unique cases up to isomorphism, where the spatial curvature k is constant; and $k > 0, k = 0, k < 0$ respectively:

$$ds^2 = -dt^2 + a(t)\begin{cases} d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2) \\ dx^2 + dy^2 + dz^2 \\ d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta d\phi^2) \end{cases}$$

Geometrically, these respectively represent space as 3-spheres, $\mathbb{R}^3$, and three-dimensional hyperboloids. The former is particularly interesting as its manifold is compact; the universe can be said to be finite.

The quantity a(t) is particularly important; it represents the 'size' of the universe. By making a (general) assumption on the form of T as a function of energy density ρ, from **1** we derive the Friedmann equations:

$$3(\frac{\dot{a}}{a})^2 = 8\pi\rho - 3\frac{k}{a^2} + \Lambda \quad (2)$$

$$3\frac{\ddot{a}}{a} = -4\pi(\rho + 3P(\rho)) + \Lambda \quad (3)$$

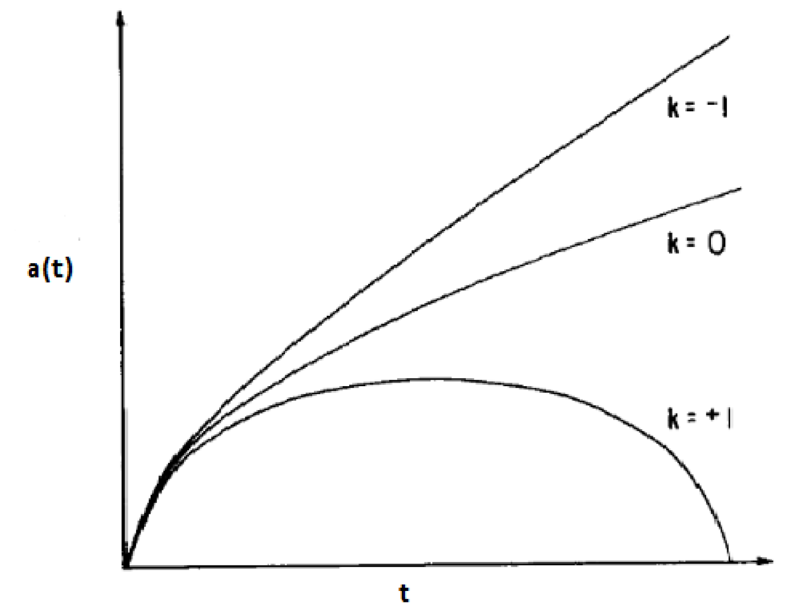

Figure 1: Qualitative behaviour of Friedmann equations for $P = 0$ and $P = \rho/3$. Via [1]

de Sitter Spacetime

de Sitter spactime is an especially interesting example of a Robertson-Walker spacetime; it is a non-trivial solution to **1** for which $T_{\alpha\beta} = 0$; i.e. the universe is without mass or energy. Yet, the solution indicates a universe with positive curvature, with metric:

$$ds^2 = -dt^2 + \alpha^2\cosh(\frac{t}{\alpha})(d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2))$$

Indicating $a(t) \to \infty$ as $|t| \to \infty$; the 'size' of space increases to ∞, despite the complete lack of energy; a disturbing result of General Relativity for many.

Future Research

Non-homogeneous solutions, while much more difficult to formulate, are more exotic than solutions shown here; offering bounds of mathematical interest.

Questions of stability of universal models and other cosmological phenomena under perturbations are being deeply researched, requiring considerations of the norm used to consider perturbations. See [3].

References

[1] M. Wald Robert. *General Relativity*. University of Chicago Press, 2nd edition, 1984.

[2] W. Hawking S. and F.R. Ellis G. *The large scale structure of space-time*. Cambridge University Press, 1974.

[3] Fournodavlos G., Rodnianski I., and Speck J. Stable big bang formation. 2020.

[4] L. I. Kharbediya. Friedmann equations with the λ-term. 1984.