The Hilbert space of the $SU(2)$ WZW model

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Introduction

Conformal field theories (CFTs) are a particular type of quantum field theory (QFT) that study the behavior of physical systems under **conformal**, or angle-preserving, transformations. In practice, 2D CFTs have been remarkably successful in describing critical phenomena, which for example occur in second order phase transitions, when a system's properties change dramatically as it approaches a critical point. CFT also has wide ranging applications in other areas of mathematics such as in string theory, a possible candidate for the theory of quantum gravity.

This goal of this poster is to write down the Hilbert space of a specific type of CFT, namely the $SU(2)$ WZW model.

Conformal invariance in 2D

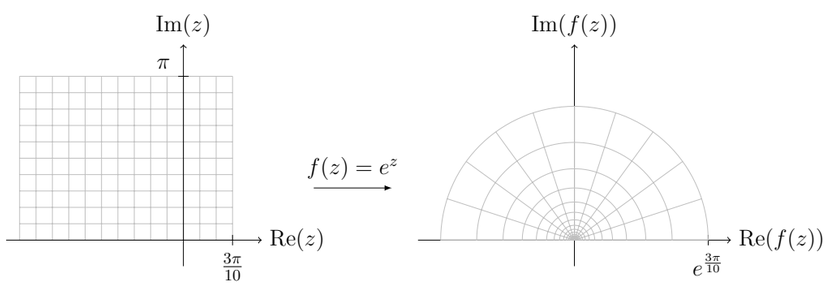

Figure 1. An example of a conformal transformation on the complex plane.

In 2D Euclidean space, conformal invariance gives rise to an infinite dimensional **symmetry algebra** which generates the local conformal transformations on the plane. In a quantum theory, this is two commuting copies of the **Virasoro algebra**, each generated by $\{L_n, c\}$ and $\{\overline{L}_n, \overline{c}\}$ ($n \in \mathbb{Z}$), with nontrivial commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$
$$[\overline{L}_m, \overline{L}_n] = (m-n)\overline{L}_{m+n} + \frac{\overline{c}}{12}(m^3 - m)\delta_{m+n,0}$$

Per the discussion of the next section, we call these copies *holomorphic* and *antiholomorphic*.

CFTs may exhibit extra symmetries, so their symmetry algebra could be a larger algebra containing the two Virasoro algebras. They still decompose into holomorphic and antiholomorphic copies as above.

Conformal fields

Conformal fields $\phi(x^\mu)$ are mathematically modelled as functions of space-time, which in two dimensions reads $\phi(x^\mu) = \phi(x,t)$. Classically, they are real valued and live on a cylinder. To transform to the complex plane we put $t = i\tau$ and make a specific change of variables:

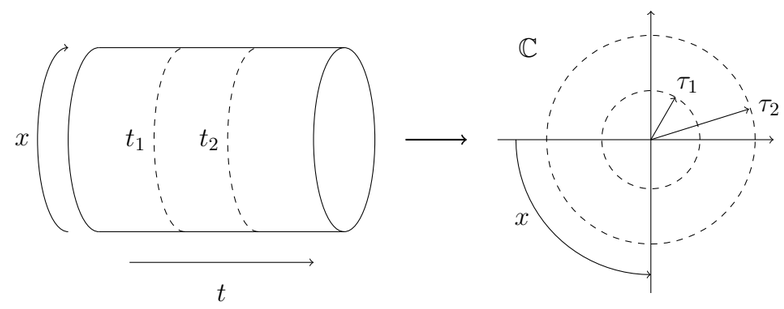

Figure 2. The transformation from the cylinder to the plane

This gives $\phi(x,t) = \phi(z,\overline{z})$. In the quantised theory, these fields admit Laurent expansions

$$\phi(z,\overline{z}) = \sum_{n,m\in\mathbb{Z}} \phi_{n,m} z^{-n-h_\phi}\, \overline{z}^{-m-\overline{h}_\phi}$$

The coefficients $\phi_{n,m}$ are referred to as the **modes** of $\phi(z,\overline{z})$ and $(h_\phi, \overline{h}_\phi) \in \mathbb{R}^2$ are called its **conformal dimensions**.

It turns out that the fields $\phi(z,\overline{z})$ which generate the symmetry algebra in a CFT decompose into holomorphic and antiholomorphic components $\phi(z)$ and $\overline{\phi}(\overline{z})$, with modes ϕ_n and $\overline{\phi}_m$ that satisfy the algebra. This produces its two commuting sectors. Conformal fields in general often follow this same factorisation.

In much of the following discussion we only consider the holomorphic sector and fields $\phi(z)$, however all results hold analogously for their antiholomorphic counterparts.

Operator product expansions & radial ordering

In a CFT, the information about how any two fields interact is encoded in an **operator product expansion** (OPE):

$$\mathcal{R}\{A(z)B(w)\} = \sum_{n\in\mathbb{Z}} \frac{C_n(w)}{(z-w)^n}$$

The $C_n(w)$ are themselves fields, and the notation $\mathcal{R}\{\cdot,\cdot\}$ refers to the **radially−ordered** product of two fields:

$$\mathcal{R}\{A(z)B(w)\} = \begin{cases} A(z)B(w) & \text{if } |z| < |w| \\ B(w)A(z) & \text{if } |w| < |z| \end{cases}$$

Importantly, an OPE of two fields is equivalent to the commutation relations of their modes (and hence the symmetry algebra):

$$[a_n, b_m] = \oint_0 \oint_w \mathcal{R}\{A(z)B(w)\}\, z^{n+h_A-1} w^{m+h_B-1} \frac{\mathrm{d}z}{2\pi i}\frac{\mathrm{d}w}{2\pi i}$$

The universal vacuum module

To extract useful information from a CFT, we need to know how fields act on vector spaces. To do this, the generators a_n of the holomorphic sector of a symmetry algebra are split into three types:

(i) a_n with $n > 0$ are called *annihilation* operators

(ii) a_n with $n = 0$ are called *zero modes*

(iii) a_n with $n < 0$ are called *creation* operators.

Using these, we define a vector $|0\rangle$ called the **vacuum** such that $a_n|0\rangle$ for all $n \geq 0$. The **universal vacuum module** V_0 is then generated from $|0\rangle$ by acting on it with strings of creation operators:

$$V_0 = \mathrm{Span}_{\mathbb{C}}\left\{\cdots a_{-3}^{n_3} a_{-2}^{n_2} a_{-1}^{n_1}|0\rangle \;\middle|\; n_i \geq 0 \text{ and } \textstyle\sum_{i=1}^{\infty} n_i < \infty\right\}$$

V_0 is an example of a module of this sector of the symmetry algebra. A field $\phi(z)$ then acts on a vector $|v\rangle \in V_0$ by $\phi(z)|v\rangle$, which can be computed explicitly via its Laurent expansion.

The state-field correspondence & quantum state space

The fields of a CFT can be used to define the **quantum state space**, or Hilbert space, $\mathcal{H}$ of the theory. Each state $|\phi\rangle \in \mathcal{H}$ is defined by the action of a field $\phi(z,\overline{z})$ on $|0\rangle \otimes |0\rangle$:

$$|\phi\rangle = \lim_{z,\overline{z}\to 0} \phi(z,\overline{z})|0\rangle \otimes |0\rangle$$

This is the **state−field correspondence**. The tensor product accounts for the antiholomorphic sector of the theory, which behaves identically to and independently of the holomorphic sector. $\mathcal{H}$ will therefore decompose (for the theory considered in this poster) into a direct sum of tensor products of irreducible modules $\mathcal{R}$ of the two sectors of the symmetry algebra, giving $\mathcal{H}$ to be of the form

$$\mathcal{H} = \bigoplus_{\mathcal{R}} \mathcal{R} \otimes \mathcal{R}$$

In general this could be much more complicated.

$\mathcal{H}$ describes all the possible states of the CFT, and contains almost all the information we care to know about the theory. It may be used to obtain the partition function of the system, thereby all its relevant physical thermodynamic quantities. Moreover, knowing the representation theory of the symmetry algebra is a crucial step in solving the theory (computing all the correlation functions).

The $SU(2)$ WZW model

In this poster, we use the Lagrangian formalism to define our field theory. To that end, consider a QFT on the compactified cylinder $\mathcal{C}$, homeomorphic to the Riemann sphere S^2. The fields of the theory are thought of as maps $g(z,\overline{z})$ from S^2 to $SU(2)$, which, topologically, is equivalent to the 3-sphere S^3:

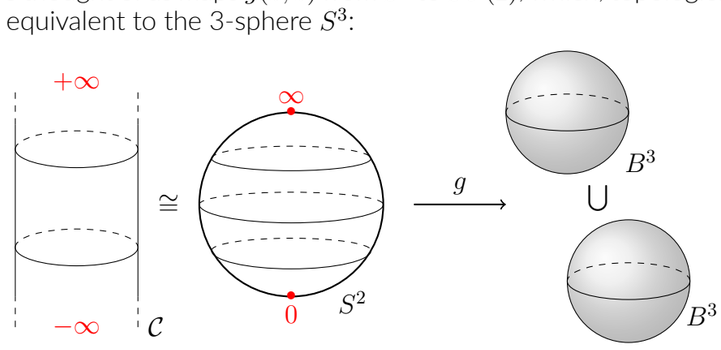

Figure 3. A visualisation of the map $g : S^2 \to SU(2) \cong S^3$. The 3-sphere S^3 is homeomorphic to two 3-balls B^3 with their boundaries identified, as above.

The model itself is defined by the action

$$S[g] = -\frac{k}{16\pi}\int_{\partial\mathcal{M}} \mathrm{Tr}\left(g^{-1}\partial_\mu g \cdot g^{-1}\partial^\mu g\right)\mathrm{d}^2x - \frac{ik}{24\pi}\int_{\mathcal{M}} \varepsilon_{abc}\,\mathrm{Tr}\left(g^{-1}\partial^a g \cdot g^{-1}\partial^b g \cdot g^{-1}\partial^c g\right)\mathrm{d}^3y$$

where k is some positive integer and $\partial\mathcal{M} = S^2$. The conserved currents can be found using standard Lagrangian methods:

$$J(z) = -k\partial_z g \cdot g^{-1}, \quad \overline{J}(\overline{z}) = kg^{-1}\partial_{\overline{z}} g$$

They take values in the complexified Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ of $SU(2)$, so expanding the holomorphic current in the canonical basis $\{E, H, F\}$ of $\mathfrak{sl}_2(\mathbb{C})$, the fields become $E(z)$, $H(z)$ and $F(z)$. Their modes $\{E_n, H_n, F_n : n \in \mathbb{Z}\}$ then comprise the basis of the symmetry algebra associated with this WZW model. Their OPEs can be calculated,

$$H(z)E(w) \sim \frac{2E(w)}{z-w}, \qquad H(z)F(w) \sim \frac{-2F(w)}{z-w},$$
$$H(z)H(w) \sim \frac{2k}{(z-w)^2}, \quad E(z)F(w) \sim \frac{k}{(z-w)^2} + \frac{H(w)}{(z-w)}$$

from which the nontrivial commutation relations of (the holomorphic sector of) the symmetry algebra can be computed:

$$[H_m, E_n] = 2E_{m+n} \qquad [F_m, H_n] = 2F_{m+n}$$
$$[H_m, H_n] = 2mk\delta_{m+n,0} \qquad [E_m, F_n] = H_{m+n} + mk\delta_{m+n,0}$$

This is the affine Kac-Moody algebra, denoted $\widehat{\mathfrak{sl}}(2)_k$. k is called the *level* of the theory. The antiholomorphic sector behaves identically, so the symmetry algebra of this CFT is two commuting copies of $\widehat{\mathfrak{sl}}(2)_k$. The Hilbert space $\mathcal{H}$ is then comprised of direct sums of tensor products of two irreducible modules of $\widehat{\mathfrak{sl}}(2)_k$. It can be checked that these generators also give rise to the required Virasoro algebras, hence the theory is indeed conformally invariant.

Highest weight modules

In the context of $SU(2)$ (or from now on $\widehat{\mathfrak{sl}}(2)_k$) WZW models, the vacuum vector $|0\rangle$ is defined to satisfy $H_n|0\rangle = E_n|0\rangle = F_n|0\rangle = 0$ for all $n \geq 0$ and V_0 is built from the creation operators F_{-n}, H_{-n}, and E_{-n} with $n > 0$. This module is an example of a **highest weight module** V_λ, which are in general constructed by acting with creation operators on a highest weight vector $|\lambda\rangle$, which must satisfy

$$H_0|\lambda\rangle = \lambda|\lambda\rangle \quad \text{and} \quad H_n|\lambda\rangle = E_{n-1}|\lambda\rangle = F_n|\lambda\rangle = 0 \quad \forall n \geq 1$$

These are the modules that appear in the expression for $\mathcal{H}$. The H_0-eigenvalue is called the 'weight' of a vector. A priori, the weights λ of these highest weight vectors could take values anywhere in $\mathbb{R}$. The aim now is to determine which of these are actually valid in the $\widehat{\mathfrak{sl}}(2)_k$ WZW model theory. To do this, we introduce the concepts of null and singular vectors.

Singular & null vectors

Physically, observables in quantum mechanics can be expressed in terms of scalar products. If a vector is orthogonal to every other state with respect to this product, then any observable quantity involving it will be zero. We call such a vector a **null vector**.

A **singular vector** $|\chi\rangle$ in a highest weight module V_λ is a descendent of $|\lambda\rangle$ that is itself a highest weight vector. Singular vectors and all of their descendants are null vectors, so guided by the motivation from quantum mechanics, they generate submodules $\langle|\chi\rangle\rangle$ of V_λ that we can then quotient by to obtain the irreducible modules $\mathcal{L}_\lambda$ which form the Hilbert space of the theory. This amounts to declaring that all the modes of the fields corresponding to the null vectors will act as the zero operator on all other vectors in the theory.

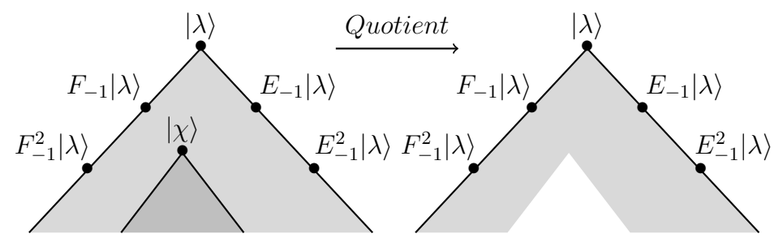

Figure 4. Quotienting V_λ (left) by the submodule $\langle|\chi\rangle\rangle$ to form $\mathcal{L}_\lambda$ (right)

Constraint on allowable highest weight vectors

We can now constrain the values of λ for an $\widehat{\mathfrak{sl}}(2)_k$ WZW model.

We begin by noting $|\psi\rangle = E_{-1}^{k+1}|0\rangle$ is a singular vector in the universal vacuum module. Since all the vectors inside the submodule $\langle|\psi\rangle\rangle$ are null, all the modes of the corresponding fields will act as the zero operator on all states. One such null vector is

$$|\chi\rangle = F_0^{2k+2}|\psi\rangle = (-1)^{k+1}(2k+2)!F_{-1}^{k+1}|0\rangle$$

Using the state-field correspondence we find the zero mode of the field $\chi(z)$ (after normalisation) and act with it on an arbitrary highest weight vector $|\lambda\rangle$:

$$\chi_0|\lambda\rangle = \sum_{n_1,\ldots,n_k\in\mathbb{Z}} F_{-\sum_{j=1}^k n_j} \prod_{j=1}^k F_{n_j}|\lambda\rangle$$

We notice that the above expression vanishes by default whenever n_j is nonzero and recall that χ_0 must be the zero operator, which leaves us with $F_0^{k+1}|\lambda\rangle = 0$. Finally, we act on this expression with E_0^{k+1} which ultimately gives us

$$E_0^{k+1}F_0^{k+1}|\lambda\rangle = (k+1)!\prod_{j=0}^{k}(\lambda - j)|\lambda\rangle = 0$$

implying that for the existence of the singular vector $|\psi\rangle$ to be consistent with the theory, we require $\lambda \in \{0, 1, \ldots, k\}$. Therefore, there are only $k+1$ allowable highest weight states in an $\widehat{\mathfrak{sl}}(2)_k$ WZW model, each characterised by a weight $\lambda \in \{0, 1, \ldots, k\}$.

The Hilbert space

In each module V_λ there is a singular vector $E_{-1}^{k+1-\lambda}|\lambda\rangle$. These singular vectors turn out to generate all the possible null states in each highest weight module, so the submodules that they generate are maximal in V_λ. It follows that the irreducible modules appearing in the expression of $\mathcal{H}$ are given by the quotient

$$\mathcal{L}_\lambda = \frac{V_\lambda}{\left\langle E_{-1}^{k+1-\lambda}|\lambda\rangle\right\rangle}$$

This finally yields the full Hilbert space of the $\widehat{\mathfrak{sl}}(2)_k$ WZW model:

$$\mathcal{H} = \bigoplus_{\lambda=0}^{k} \mathcal{L}_\lambda \otimes \mathcal{L}_\lambda$$

References

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[3] David Ridout. An introduction to conformal field theory, April 2013.

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