# Partition functions of six-vertex models 

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## Introduction

The six-vertex model was first introduced in the early 20th century in statistical mechanics as a means to describe crystal lattices with hydrogen bonds. Now, it features prominently in key areas of mathematical research, such as

- integrable models
- quantum stochastic processes
- enumerative combinatorics

In this project, we focus on the construction of the partition functions of six-vertex models under certain parameterisations

## Construction of the Lattice

We consider the six-vertex model in two dimensions, where it is a directed square lattice. At any given edge, a 1 indicates a path, and a 0 indicates the absence of a path. Allowed vertices are pathpreserving, meaning 0 s and 1 s entering on the left and bottom must equal those exiting through the right and top. This results in six valid configurations, shown with their weights below:


1. Spectral weights

The spectral parameter $z$ is calculated in Figure 1, as the ratio of the incoming and outgoing variables.
We also consider a lattice that features 'corner' weights. These have Boltzmann weights and pictorial representation:


Where $p, q$ and $r$ are free parameters, $x$ is the incoming parameter, and

$$
h(x)=\frac{p r\left(1-x^{2}\right)}{(p-x)(r-x)}
$$

The weight of a particular lattice is given by the product of all vertex Boltzmann weights on the lattice. The partition function is calculated as the sum of weights over all possible configurations of a given lattice.
We consider two lattices under set boundary conditions and their associated partition functions: the domain wall partition function $Z_{m}$;

and the triangular partition function $T_{m}$


Under our choice of Boltzmann weights, it can be shown that $Z_{m}$ will be a rational function in each $x_{i}$ with degree $m-1$ numerator and denominator, and $T_{m}$ a rational function with degree $m+1$ numerator and denominator

## The R-Matrix and K-Matrix

We can store our vertex and corner weights in algebraic objects called the $R$-matrix and $K$-matrix, which have representation:

$$
R(z)=\left(\begin{array}{cccc}
a(z) & 0 & 0 & 0 \\
0 & b(z) & c(z) & 0 \\
0 & c(z) & b(z) & 0 \\
0 & 0 & 0 & a(z)
\end{array}\right) \quad K(x)=\left(\begin{array}{cc}
t_{1}(x) & t_{2}(x) \\
t_{3}(x) & t_{4}(x)
\end{array}\right)
$$

Where $a, b$ and $c$ denote the Boltzmann weights of the $\mathrm{a}-\mathrm{b}$ - and ctype vertices, and the entries of the $K$-matrix are the corner weights. These matrices can be used combinatorially with the Kronecker product to produce matrices whose entries correspond to certain partition functions under particular boundary conditions.

## Identities on the Lattice

Our choice of Boltzmann weights results in a number of powerful identities which we can exploit to probe the structure of our partition function. Our weights permit a form of the Yang-Baxter equation, which has pictorial and algebraic representation:
$\left(R_{1,2} \otimes \mathbb{1}\right)\left(\mathbb{1} \otimes R_{1,3}\right)\left(R_{2,3} \otimes \mathbb{1}\right)=\left(\mathbb{1} \otimes R_{2,3}\right)\left(R_{1,3} \otimes \mathbb{1}\right)\left(\mathbb{1} \otimes R_{1,2}\right)$
And for corner weights, the analogous reflection equation:


Which may be used to simplify sections of our lattices.

## Symmetry of the Lattice

Under this choice of weights, we may show that $Z_{m}$ is a symmetric function in $\left\{x_{i}\right\}$ and $\left\{y_{i}\right\}$. We first introduce a crossing at one edge of the lattice, which does not alter the value of the partition function as it must be a-type and have weight 1 under our boundary conditions This crossing may be brought through the lattice using the YangBaxter relation as follows:

We then freely remove the crossing, as it has weight 1 .
The above shows we may freely interchange any adjacent $x_{i} \leftrightarrow x_{i+1}$ or $y_{i} \leftrightarrow y_{i+1}$, and consequently permute these variables amongst themselves without altering our partition function

The same argument can be used to show that $T_{m}$ is a symmetric function in $\left\{x_{i}\right\}$, where the reflection equation is utilised to bring a crossing over a corner pair.

## Recursions on the Lattice

We can also see that both partition functions $Z_{m}$ and $T_{m}$ satisfy a number of recursion relations, by properties of the lattice. For brevity, we only sketch a proof of two key 'specialisations'. See Figure 2 for the exhaustive list.

## One-step recursion on $Z_{m}$

We consider the specialisation $x_{i} \rightarrow y_{j}$. As $Z_{m}$ is symmetric, we may translate $x_{i}$ and $y_{j}$ such that their intersection occurs at the bottomleft of the lattice. Note that the spectral weight of this vertex $z=$ $x_{i} / y_{j}$ is 1 under this specialisation, so we utilise unitarity of $R$ to remove the vertex. Our boundary conditions force the remaining vertices along the bottom row and left column to be a-type



All forced vertices contribute a joint Boltzmann weight of 1, and translate the same domain wall boundary conditions into the ( $m$ 1) $\times(m-1)$ sub-lattice. Hence we may discard of $x_{i}$ and $y_{j}$, and find the recursion relation:

$$
\left.Z_{m}(\vec{X} ; \vec{Y})\right|_{x_{i}=y_{j}}=Z_{m-1}\left(\vec{X} \backslash\left\{x_{i}\right\} ; \vec{Y} \backslash\left\{y_{j}\right\}\right)
$$

## Two-step recursion on $T_{m}$

Where $i \neq j$, consider the specialisation $x_{i} \rightarrow x_{j}^{-1}$. By the symmetry of $T_{m}$, we may consider $x_{2} \rightarrow x_{1}^{-1}$ without loss of generality. This sends the spectral weight at the intersection to 1 , which can then be removed by $R$-unitarity. The bottom row vertices are forced as a-type, and we use the unitarity property of $K$ to remove the two corner weights. This now forces all vertices in the second-bottom row to be a-type as well.


Again, we discard of these two rows without altering the partition function, and find the $(m-2) \times(m-2)$ sub-lattice with identical boundary conditions. Hence we have the recursion relation:

$$
\left.T_{m}(\vec{X})\right|_{x_{i}=x_{i}^{-1}}=T_{m-2}\left(\vec{X} \backslash\left\{x_{i}, x_{j}\right\}\right)
$$

## The Partition Functions

We find expressions for our partition functions;

$$
\begin{gather*}
Z_{m}(\vec{X} ; \vec{Y})=\frac{(1-q)^{m} \prod_{i=1}^{m} x_{i}^{\frac{1}{2}} y_{i}^{\frac{1}{2}} \prod_{i, j=1}^{m}\left(x_{i}-y_{j}\right)}{\prod_{1 \leq i<j \leq m}\left(x_{i}-x_{j}\right)\left(y_{j}-y_{i}\right)} \operatorname{det}\left[\frac{1}{x_{i}-y_{j}} \frac{1}{x_{i}-q y_{j}}\right]  \tag{*}\\
T_{m}(\vec{X})=\prod_{i=1}^{m} x_{i}^{-\frac{1}{2}} \prod_{1 \leq i<j \leq m} \frac{1-x_{i} x_{j}}{x_{i}-x_{j}} \times \operatorname{Pf}\left[\frac{x_{i}-x_{j}}{1-x_{i} x_{j}} Q\left(x_{i}, x_{j}\right)\right] \tag{**}
\end{gather*}
$$

Where the function $Q$ is symmetric in its arguments and given by:

$$
Q\left(x_{i}, x_{j}\right)=\left(\left(1-h\left(x_{i}\right)\right)\left(1-h\left(x_{j}\right)\right)-h\left(x_{i}\right) h\left(x_{j}\right) \frac{(1-q) x_{i} x_{j}}{p r\left(1-q x_{i} x_{j}\right)}\right)
$$

And Pf denotes the Pfaffian, which is formally defined:

$$
\operatorname{Pf}(A)=\frac{1}{2^{n} n!} \sum_{\sigma \in S_{2 n}} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(2 i-1), \sigma(2 i)}
$$

Where A is a skew-symmetric matrix $\left(A^{T}=-A\right)$, and we have the relation $\operatorname{Pf}(A)^{2}=\operatorname{det}(A)$. The Pfaffian behaves similarly to a determinant -however a row operation must be done simultaneously to the corresponding columns (and vice-versa) in order to preserve the skew-symmetry of $A$.
The Pfaffian also has a recursive definition similar to that of Laplace expansion

$$
\operatorname{Pf}(A)=\sum_{\substack{j=1 \\ j \neq i}}^{2 n}(-1)^{i+j+1+\theta(i-j)} a_{i j} \operatorname{Pf}\left(A_{\hat{\imath}}\right)
$$

Where $\theta$ is the Heaviside step function and $A_{\hat{\jmath} \hat{\jmath}}$ denotes the matrix $A$ with both columns and rows $i$ and $j$ removed.

|  | Specialisation | Result |
| :--- | :--- | :--- |
| $Z_{m}$ | $x_{i} \rightarrow y_{j}$ | $Z_{m-1}\left(\vec{X} \backslash\left\{x_{i}, y_{j}\right\}\right)$ |
|  | $x_{i} \rightarrow 0$ | 0 |
|  | $x_{i} \rightarrow x_{j}^{-1}$ | $T_{m-2}\left(\vec{X} \backslash\left\{x_{i}, x_{j}\right\}\right)$ |
|  | $x_{i} \rightarrow 1$ | $T_{m-1}\left(\vec{X} \backslash\left\{x_{i}\right\}\right)$ |
|  | $x_{i} \rightarrow-1$ | $-i T_{m-1}\left(\vec{X} \backslash\left\{x_{i}\right\}\right)$ |
|  | $x_{i} \rightarrow 0$ | 0 |

Figure 2. Specialisations for $Z_{m}$ and $T_{m}$

## Construction of the Partition Functions

It turns out that the recursion relations in Figure 2 uniquely define our partition functions. Hence, the expressions in ( $*$ ) and ( $* *$ ) are not derived, but constructed to meet these specialisations. We can, however, motivate their construction.

For instance, the determinant in $Z_{m}$ is a natural method of encoding the recursion relation in (1). Taking $x_{m} \rightarrow y_{m}$ leads to a zero in the product term of $(*)$ :

$$
\prod_{i, j=1}^{m-1}\left(x_{i}-y_{j}\right) \times\left(x_{m}-y_{m}\right)=0
$$

But creates a singularity in the ( $m, m$ ) entry of our determinant:

$$
\operatorname{det}\left[\because \frac{1}{x_{m}-y_{m}} \frac{1}{x_{m}-q y_{m}} \ddots\right] \rightarrow \infty
$$

Through cofactor expansion, the zero appearing in the product causes all terms to vanish, except for this $(m, m)$ term where the singularity and zero cancel. We are then left with a smaller $(m-1) \times(m-1)$ determinant with the $m^{\text {th }}$ row and column removed, as desired.
Similarly, the Pfaffian in $T_{m}$, when expanded using ( $\dagger$ ), encodes the two-step recursion in (2), where all minors vanish except that with the $i^{\text {th }}$ and $j^{\text {th }}$ rows and columns removed.

## Conclusion

In the construction of these partition functions, we have relied heavily on the Yang-Baxter equation and analogous relations. These tools originate in the study of integrable systems, and are incredibly powerful for constructing exact solutions to mathematical problems. Our partition functions have numerous further applications:

- evaluation at a key 'ice point' gives the enumeration of a complex class of matrices, known as alternating sign matrices
- the joint square and triangular lattice (Figure 3) is related to a well-studied Markov process known as the asymmetric simple exclusion principle, which the partition functions help to analyse


Figure 3. The adjoined triangular and square lattice

## References

[^0]
[^0]:    1] Alexandr Garbali, Jan de Gier, William Mead, and Michael Wheeler Symmetric function
    2] Greg Kuperberg.
    Symme
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