Solutions: wave dispersion relation in porous water layers

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Introduction

Wave dispersion relation is essential in **fluid dynamics**. It describes how waves of different frequencies travel at different speed and how they affect porous medium (reef) and water layer. Waves traveling over porous media, such as coral reefs, seabeds, have behaviors significantly different from those in open water and affect momentum transfer within the porous layer. The wave dispersion relation provides a powerful mathematical framework to quantify how wave frequency and wavenumber are related in such layered systems. Investigating these dynamics is essential for predicting wave attenuation and stability of marine environments.

- Deriving wave dispersion relation $k(\omega)$ involves **PDE** with dynamic and kinematic boundary conditions.
- Consider solutions to dispersion relation in the limit of no porous layer
- How the solutions change when considering porous layer.
- How can the model be extended to the case when **porous layer is on top** of water layer, like in Figure 2 ?

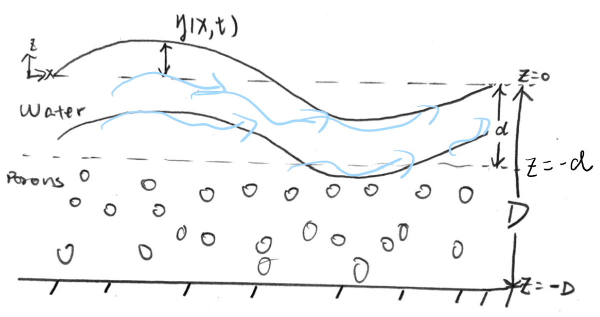

Figure 1. Water layer on top

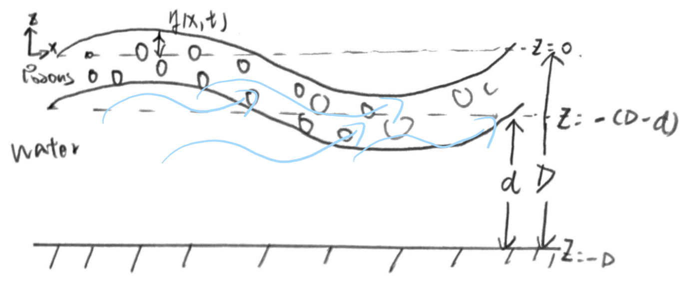

Figure 2. Porous layer on top

No porous layer solutions

Consider limiting case with no porous layer, only the inviscid and incompressible fluid of finite depth above porous medium.

Assume that **free surface elevation** is $\eta(x,\ t) = \mathrm{Re}[Ae^{\mathrm{i}(kx-\omega t)}]$, k can be **complex**.

$\phi(x, z, t)$ is the **velocity potential** of water layer Consider the material derivative on free surface $\frac{D}{dt}(z - \eta(x,\ t)) = 0$ which is

$$-\frac{\partial \eta}{\partial t} + \nabla\phi \cdot \left(\frac{\partial \eta}{\partial x},\ 1\right) = 0$$

Boundary Conditions after linearisation

$$\text{dynamic BC:}\quad \frac{\partial\phi}{\partial z} = \frac{\partial \eta}{\partial t} \quad \text{at } z = 0$$

$$\text{kinematic BC:}\quad \frac{\partial\phi}{\partial t} + g\eta = 0 \quad \text{at } z = \eta$$

By solving $\nabla^2\phi = 0$ with the assumption of the wave form, the velocity potential of water layer becomes

$$\phi(x,\ z,\ t) = C_1 \cosh\,(kz + C_2)e^{\mathrm{i}(kx-\omega t)}$$

Combining the above equations of k and ω, the classical dispersion relation of no porous layer is

$$\omega^2 = gk\tanh\,[kd]$$

for a fixed real frequency ω, this relation is solved for wavenumber k from the contour plot as in Figure 3 where solutions are marked with red dots, there is a **unique real root**, associated with wave propagation, and infinitely **many imaginary roots**, associated with **evanescent modes**.

For the first branch, $\mathrm{Re}(k)$ increases as $\omega \to \infty$ while $\mathrm{Im}(k) = 0$, representing propagating waves. For the higher branch, $\mathrm{Im}(k)$ increases as $\omega \to \infty$ and asymptotically approaches a finite value, representing decaying mode due to presence of $e^{\mathrm{i}(kx-\omega t)}$.

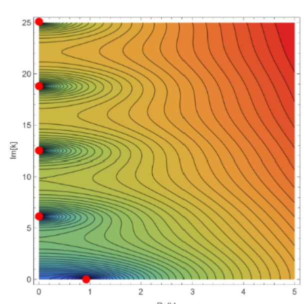

Figure 3. Contour plot

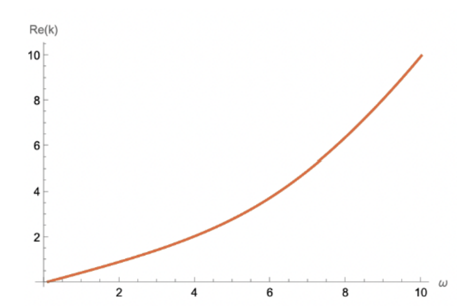

Figure 4. Re(k) vs ω

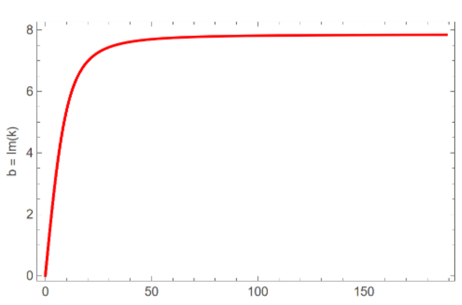

Figure 5. Im(k) vs ω

Water-porous model calculations

Consider with porous layer at the bottom as in Figure 1, and solving 2D Laplace's equation $\nabla^2 p = 0$ where $p(x, z, t)$ is the pressure field within porous layer. With the assumption that

$$p(x,\ z,\ t) = \tilde{p}(z)e^{\mathrm{i}(kx-\omega t)} + C_3$$

which can further reduce the partial differential equation to second order, homogeneous, linear **ordinary differential equation**:

$$\frac{d^2\tilde{p}}{dz^2} - k^2\tilde{p} = 0$$

and imposing the condition that $d\tilde{p}/dz = 0$ at the impermeable bottom $z = -D$. The pressure field becomes

$$p(x,\ z,\ t) = C_3\ +\ C_4\cosh\,[k(z + D)]e^{\mathrm{i}(kx-\omega t)}$$

Furthermore, by continuity of vertical velocity and pressure at the interface. Combining the equations, the dispersion relation connects surface wave frequency ω and complex wavenumber k in

$$-\Pi(-d)C_4\sinh\,[k(D-d)] = \mu C_1\sinh\,(C_2 - kd)$$

Let the dimensionless constant $J = \rho\omega\Pi(-d)/\mu$ and rewrite the dispersion relation in a function of k and ω, $f(k,\ \omega; J,\ D,\ d,\ g)$ where J, D, d, g are parameters.

Porous layer solutions

Consider the Original case that water layer is on top of porous layer with finite depth.
Consider with porous layer, solving PDE for $p(x, z, t)$ and matching two layers by **continuity of vertical velocity** and **pressure** at the interface:

$$f(k,\ \omega; J,\ D,\ d,\ g) = J\tanh\,[k(D-d)] - \mathrm{i}\tanh\,[\operatorname{arctanh}(\tfrac{\omega^2}{gk}) - kd]$$

With parameters set at $d = 0.5$, $J = 1$, $D = 1$, $g = 10$. Solving $f(k,\ \omega) = 0$ gives complex wave number $k = k_r + \mathrm{i}\,k_i$.

Solutions are marked by red dots in the contour plot Figure 6. The solution branch with relatively large Real Parts and small Imaginary Parts are **propagating solution**.

Other solutions are $k_r \ll k_i$ which are **evanscent modes** that decay. The presence of these solutions highlights the interaction between the water layer depth and the porous medium in shaping the wave dynamics. **Note:** for distinction with above case, using z to represent wavenumber in the plots here.

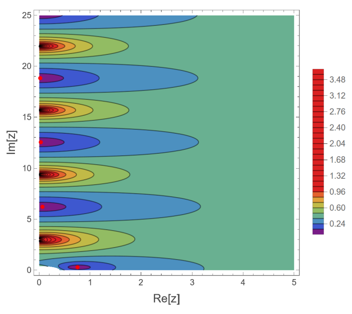

Figure 6. Contour plot

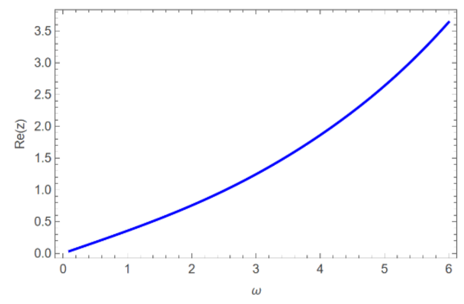

Figure 7. Re(z) vs ω

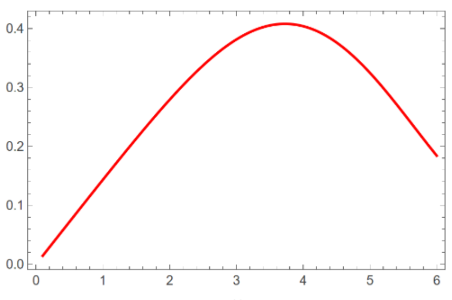

Figure 8. Im(z) vs ω

Brief extension of model: porous layer on top

Consider the case that porous layer is on top of water layer with finite depth and layer thickness as in Figure 2. Assume the fluid is **inviscid, incompressible and irrotational**.

Flow within porous layer

For the porous medium with isotropic permeability $\Pi(z)$, which is a function of z, from Darcy's law, the volumetric flux $\boldsymbol{q}$ is given by

$$\boldsymbol{q} = -\frac{\Pi(z)}{\mu}\boldsymbol{\nabla}p$$

where μ is the dynamic viscosity, p is the pressure field within porous medium with assumptions that $p(x, z, t) = \tilde{p}(z)e^{\mathrm{i}(kx-\omega t)}$ and let $\boldsymbol{q} = (u, w)$. At the free surface $z = \eta(x, t)$, dynamic boundary condition indicates

$$w = \frac{\partial\eta}{\partial t} + u\frac{\partial\eta}{\partial x}$$

by the assumption of linear theory, it is linearised to give

$$-\frac{\Pi(0)}{\mu}\frac{\partial p}{\partial z} = \frac{\partial\eta}{\partial t}$$

at $z = 0$. Also, we can impose the kinematic boundary condition from Bernoulli equation with the assumption of zero atmospheric pressure, at $z = \eta(x, t)$,

$$p(x, \eta, t) + \frac{\rho}{2}|\boldsymbol{u}|^2 + \rho g\eta = 0$$

By incompressibility of flow within porous layer, $\boldsymbol{\nabla}\cdot\boldsymbol{q} = 0$, which is to be solved for $z > -(D-d)$, gives **linear, second order, homogeneous PDE**. This can be further reduced to **ODE** with unknown $\tilde{p}(z)$,

$$\Pi(z)\left[\frac{d^2\tilde{p}}{dz^2} - k^2\tilde{p}(z)\right] + \frac{d\Pi}{dz}\frac{d\tilde{p}}{dz} = 0$$

Flow under porous layer

At $z = -D$, the rigid seabed, BC is $\partial\phi/\partial z = 0$. With assumption of $\phi(x, z, t) = \tilde{\phi}(z)e^{\mathrm{i}(kx-\omega t)}$ gives

$$\phi(x, z, t) = C_1\cosh\,[k(z + D)]e^{\mathrm{i}(kx-\omega t)}$$

this is the velocity potential of water layer underneath.

Matching two layers

At $z = -(D-d)$, the interface of two layers.
From **continuity of vertical velocity**,

$$\frac{\partial\phi}{\partial z} = -\frac{\Pi(z)}{\mu}\frac{\partial p}{\partial z}$$

which becomes

$$kC_3\sinh\,[k(z + D)] = -\frac{\Pi(z)}{\mu}\frac{\partial\tilde{p}}{\partial z}$$

From **continuity of pressure** and governed by Navier-Stokes equation for incompressible flow

$$p = -\rho\frac{\partial\phi}{\partial t}$$

Combining the above relations about k and ω, gives the **dispersion relation** in terms of $\tilde{p}$ and its derivative

$$\frac{k\tanh\,(kd)}{\mathrm{i}\omega\rho}\,\tilde{p}(d-D) = -\frac{\Pi(d-D)}{\mu}\frac{\partial\tilde{p}}{\partial z}\Big|_{z=d-D}$$

where $\tilde{p}$ and its partial derivative are functions of k.

Acknowledgments and references

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[Webber and Huppert 2021] Webber, J. J. and Huppert, H. E., *Stokes drift through corals*, Environmental Fluid Mechanics, 21, 1119–1135 (2021).