Melbourne Uni Research Competition

Question 7:

In this problem, we are given two mappings:

$$T_1(x) = 2 - \frac{1}{x} \text{ and } T_2(x) = 1 - \frac{1}{x}$$

and their inverses:

$$T_1^{-1}(x) = \frac{1}{2-x} \text{ and } T_2^{-1}(x) = \frac{1}{1-x}$$

We must find a formula (if one exists) with these four mappings which reduces any number $\frac{a}{b}$ to 0.

Let us substitute $\frac{a}{b}$ into these mappings and assume the output is $\frac{a'}{b'}$

$T_1\left(\frac{a}{b}\right) = 2 - \frac{b}{a} = \frac{2a}{a} - \frac{b}{a} = \frac{2a-b}{a}$ so $a' = 2a - b$ and $b' = a$

$T_2\left(\frac{a}{b}\right) = 1 - \frac{b}{a} = \frac{a}{a} - \frac{b}{a} = \frac{a-b}{a}$ so $a' = a - b$ and $b' = a$

$T_1^{-1}\left(\frac{a}{b}\right) = \frac{1}{2-\frac{a}{b}} = \frac{1}{\frac{2b-a}{b}} = \frac{b}{2b-a}$ so $a' = b$ and $b' = 2b - a$

$T_2^{-1}\left(\frac{a}{b}\right) = \frac{1}{1-\frac{a}{b}} = \frac{1}{\frac{b-a}{b}} = \frac{b}{b-a}$ so $a' = b$ and $b' = b - a$

We will also work out $T_2^{-1}[T_1\left(\frac{a}{b}\right)]$ as this will help us in the future.

$T_2^{-1}\left[T_1\left(\frac{a}{b}\right)\right] = T_2^{-1}\left[\frac{2a-b}{a}\right] = \frac{1}{1-\frac{2a-b}{a}} = \frac{1}{\frac{a}{a}-\frac{2a-b}{a}} = \frac{1}{\frac{b-a}{a}} = \frac{a}{b-a}$ so $a' = a$ and $b' = b - a$

We assume that the fraction $\frac{a}{b}$ is in its simplest form, if $\frac{a}{b} < 0$, then $a < 0$ and $b > 0$, and $b \neq 0$ so the fraction is not infinite.

Let us define an iterative method which uses the following rules:

1. If a = 0 then we are done as $\frac{a}{b}$ will equal 0.
2. If $a < 0$ (so $b > 0$ and $\frac{a}{b} < 0$) then apply $T_2^{-1}\left(\frac{a}{b}\right) = \frac{b}{b-a}$.
3. If $a \geq b$ (so $\frac{a}{b} \geq 1$) then apply $T_2\left(\frac{a}{b}\right) = \frac{a-b}{a}$
4. If $\frac{1}{2} \leq \frac{a}{b} < 1$ then apply $T_1\left(\frac{a}{b}\right) = \frac{2a-b}{a}$
5. If $0 < \frac{a}{b} < \frac{1}{2}$ then apply $T_2^{-1}\left[T_1\left(\frac{a}{b}\right)\right] = \frac{a}{b-a}$

These rules should be applied to the original number, $\frac{a}{b}$ until it has been reduced to 0.

An example has been shown which will reduce $\frac{3}{7}$ to 0 using the iterative method:

Since $0 < \frac{3}{7} < \frac{1}{2}$ we must apply ${T_2}^{-1}\left[T_1\left(\frac{3}{7}\right)\right] = \frac{3}{7-3} = \frac{3}{4}$

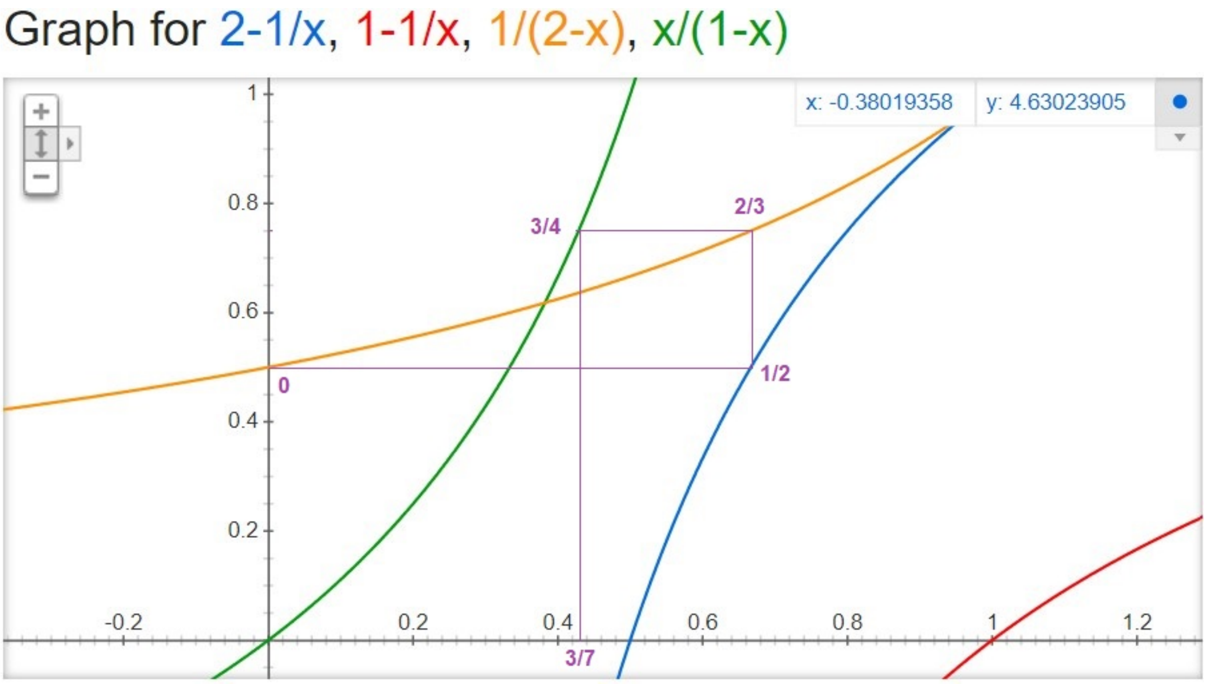
Since $\frac{1}{2} \leq \frac{3}{4} < 1$ we must apply $T_1\left(\frac{3}{4}\right) = \frac{6-4}{3} = \frac{2}{3}$

Since $\frac{1}{2} \leq \frac{2}{3} < 1$ we must apply $T_1\left(\frac{2}{3}\right) = \frac{4-3}{2} = \frac{1}{2}$

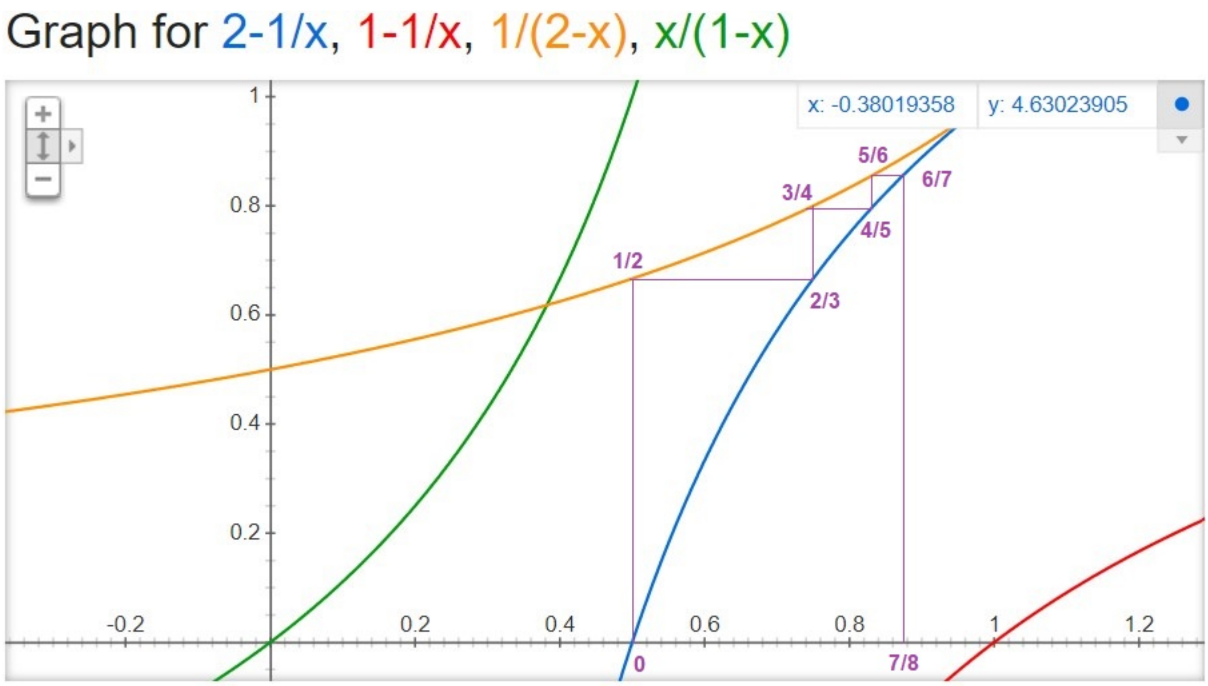
Since $\frac{1}{2} \leq \frac{1}{2} < 1$ we must apply $T_1\left(\frac{1}{2}\right) = \frac{2-2}{1} = \frac{0}{1} = 0$

Since the number is now 0, we are done and $\frac{3}{7}$ has now been reduced to 0.

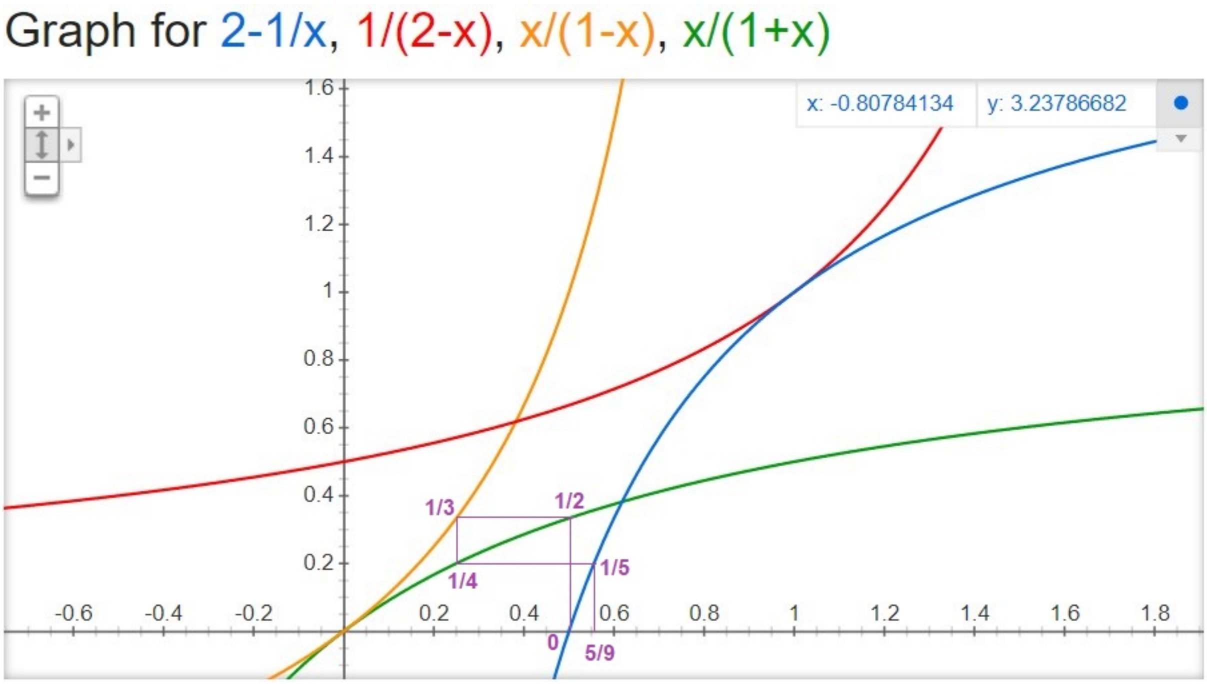
Graphs showing $\frac{3}{7}, \frac{7}{8}$ and $\frac{5}{9}$ being reduced to 0 using this method are shown below.



$\frac{3}{7}$ being reduced to 0



$\frac{7}{8}$ being reduced to 0



$\frac{5}{9}$ being reduced to 0

We will now prove why this iterative method will work for all fractions $\frac{a}{b}$.

1. If $\frac{a}{b} = 0$ then we are done as $\frac{a}{b}$ has already been reduced.

2. If $\frac{a}{b} < 0$ then we should apply ${T_2}^{-1}\left(\frac{a}{b}\right) = \frac{b}{b-a}$. Since $a < 0$ and $-a > 0, b - a > b > 0$. So, $0 < \frac{b}{b-a} < 1$. Now, we can apply Rule 4 or 5 to $\frac{b}{b-a}$ depending on whether $\frac{b}{b-a} < \frac{1}{2}$ or not.

3. If $\frac{a}{b} \geq 1$ then we should apply $T_2\left(\frac{a}{b}\right) = \frac{a-b}{a}$. Since $a \geq b, a - b \geq 0$. Also, $a - b \leq a$ so $0 \leq \frac{a-b}{a} < 1$. Either $\frac{a-b}{a} = 0$ (thus we are done) or $0 < \frac{a-b}{a} < 1$. Now, we can apply Rule 4 or 5 to $\frac{a-b}{a}$, depending on whether $\frac{a-b}{a} < \frac{1}{2}$ or not.

4. If $\frac{1}{2} \leq \frac{a}{b} < 1$ then we should apply $T_1\left(\frac{a}{b}\right) = \frac{2a-b}{a} = 2 - \frac{b}{a}$. $\frac{a}{b} \geq \frac{1}{2},\ 2a \geq b,\ 2a - b \geq 0$, also $\frac{b}{a} \leq 2,\ \frac{b}{a} - 2 \leq 0,\ 2 - \frac{b}{a} \geq 0$. Furthermore, $\frac{a}{b} < 1,\ \frac{b}{a} > 1,\ \frac{b}{a} - 1 > 0,\ \frac{b}{a} - 2 > -1,\ 2 - \frac{b}{a} < 1$. So, $0 \leq 2 - \frac{b}{a} < 1$. Now, we can apply Rule 1, 4 or 5 to $2 - \frac{b}{a}$, depending on whether $2 - \frac{b}{a} = 0, or\ < \frac{1}{2}, or > \frac{1}{2}$.

5. If $0 < \frac{a}{b} < \frac{1}{2}$ then we should apply ${T_2}^{-1}\left[T_1\left(\frac{a}{b}\right)\right] = \frac{a}{b-a}$. $\frac{a}{b} < \frac{1}{2},\ 2a < b,\ 2a - b < 0,\ b - 2a > 0,\ b - a > 0,\ \frac{a}{b-a} > 0$. Furthermore, $b - 2a > 0,\ b - a > a,\ \frac{b-a}{a} > 1,\ \frac{a}{b-a} < 1$. So, $0 < \frac{a}{b-a} < 1$. Now, we can apply Rule 4 or 5 to $\frac{a}{b-a}$, depending on whether $\frac{a}{b-a} < \frac{1}{2}$ or not.

We will now show that the iterative method will result in the number being reduced to 0 and not create an infinite loop. Steps 1,2, and 3 either result in the reduction and the iteration stops or 4 and 5 being applied. We thus focus on steps 4 and 5.

Let us define $n = b - a$.

We first consider step 4 applied when $\frac{1}{2} \leq \frac{a}{b} < 1$. $n = b - a > 0$ since $b > a$. Also $n < b$ and $n \leq a$..We apply $T_1\left(\frac{a}{b}\right) = \frac{2a-b}{a} = \frac{a-(b-a)}{b-(b-a)} = \frac{a-n}{b-n}$. This means that the numerator and denominator of the output will always be smaller than the numerator and denominator of $\frac{a}{b}$.

If we can reduce the number to $\frac{1}{2}$, we are done as we can apply $T_1\left(\frac{a}{b}\right) = \frac{2a-b}{a}$ to it to make it 0. Therefore, we want the denominator and numerator to be as small as possible.

When $\frac{1}{2} \leq \frac{a}{b} < 1$, let us calculate $\frac{a}{b} - \frac{(a-n)}{(b-n)}$.

$$\frac{a}{b} - \frac{(a-n)}{(b-n)} = \frac{[a(b-n)-(a-n)b]}{b(b-n)} = \frac{ab-an-ab+bn}{b(b-n)} = \frac{(b-a)n}{ab} = \frac{(b-a)^2}{ab} > 0$$

So, $\frac{a}{b} - \frac{(a-n)}{(b-n)} > 0,\ \frac{(a-n)}{(b-n)} < \frac{a}{b},\ \frac{a-(b-a)}{b-(b-a)} < \frac{a}{b},\ \frac{2a-b}{a} < \frac{a}{b},\ T_1\left(\frac{a}{b}\right) = \frac{2a-b}{a} < \frac{a}{b}$

Therefore, every time the fraction $\frac{a}{b}$ is applied to $T_1\left(\frac{a}{b}\right) = \frac{2a-b}{a}$ (following Rule 4) the output will be smaller than $\frac{a}{b}$. The iteration with rule 4 will continue until either $\frac{a}{b} = \frac{1}{2}$ and the fraction is subsequently reduced or $0 < \frac{a}{b} < \frac{1}{2}$ and rule 5 applies.

We now consider step 5 applied when $0 < \frac{a}{b} < \frac{1}{2}$. Let us compare $\frac{a}{b}$ and $\frac{a}{b-a}$.

$\frac{a}{b-a}$ is the output of the mapping ${T_2}^{-1}\left[T_1\left(\frac{a}{b}\right)\right] = \frac{a}{b-a}$ (when following Rule 5) and we can see that compared to $\frac{a}{b}$, the numerator is constant and the denominator decreases. The iteration with rule 5 will continue until $\frac{1}{2} \leq \frac{a}{b} < 1$and rule 4 applies.

Therefore, when applying Rule 4 and 5 of the iterative method to a fraction the numerator will either remain the same or decrease and denominator will always decrease. This means that eventually, they will converge to 0 and $\frac{a}{b}$ will be reduced to 0.

Thus, we have shown that this iterative method will reduce any number $\frac{a}{b}$ to 0.