CONWAY'S RATIONAL TANGLES AND RATIONAL NUMBERS

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Abstract. The reduction of rational numbers through a pair of functions and their inverses is analogous to the untangling of Conway's rational tangles. The representation of rational numbers as corresponding tangles through continued fractions yields a method to reduce rational numbers to 0: alternating expressions of inverse tangle operations with the functions examined form the basis of this approach. A follow up study aimed at determining a universal method for a family of functions in similar forms is also suggested.

1. Introduction

Functions, as presented on a high school level, are often studied in relation to algebra and calculus. Function composition, however, is often used to construct faithful representations of groups in abstract algebra, where the functions form right actions on corresponding groups.

The problem to be examined here was posed by the University of Melbourne, and involves the mapping of rational numbers to 0 through two functions and their inverses; we define them as follows.

Definition 1.1. Let T_1, T_2, and their inverses be functions on $\mathbb{Q}^* \to \mathbb{Q}^*$, where $\mathbb{Q}^* = \mathbb{Q} \cup \{\infty\}$, $\frac{1}{\infty} \mapsto 0$, $\frac{1}{0} \mapsto \infty$.

$$T_1(x) = 2 - \frac{1}{x} \qquad T_2(x) = 1 - \frac{1}{x}$$

$$T_1^{-1}(x) = \frac{1}{2-x} \qquad T_2^{-1}(x) = \frac{1}{1-x}$$

The prominence of function composition in the reduction of a rational number suggests a connection to abstract algebra. A closer observation of the problem yields an interesting resemblance to Conway's rational tangles; in particular, the functional representation of geometric tangle operations allows us to draw parallels between the untangling process of tangles and the reduction of rational numbers. This forms the basis of our approach: under such interpretation, we produce a method of reducing rationals that is analogous to the untangling of tangles.

2. Conway's Rational Tangles

Before we examine the applications of rational tangles to our problem, we provide a brief outline of rational tangles.

Key words and phrases. Rational numbers, Conway's rational tangles, transformations.

2.1. **A Tale of T and R.** By slicing apart knots and pinning down the cut ends, we obtain a class of mathematical objects known as *tangles.* We begin by considering two geometric operations, *Twist* and *Rotate*, that act on rational tangles.

Definition 2.1 ([2])**.** A rational tangle is any tangle which can be obtained by performing the following operations on an empty tangle, where an empty tangle (denoted by [0]) describes two horizontal, parallel strands.

(1) **"Twisting"** bottom right over top right. (T)
(2) **"Rotating"** the whole tangle 90° clockwise. (R)

Definition 2.2 (Simple Rational Tangles [1])**.**

(1) An infinity tangle, denoted by $[\infty]$, describes two vertical, parallel strands.
(2) An integer tangle, denoted by $[n]$, is made up of n horizontal twists. ($n \in \mathbb{Z}$)
(3) A vertical tangle, denoted by $\frac{1}{[n]}$, is made up of n vertical twists. ($n \in \mathbb{Z}$)

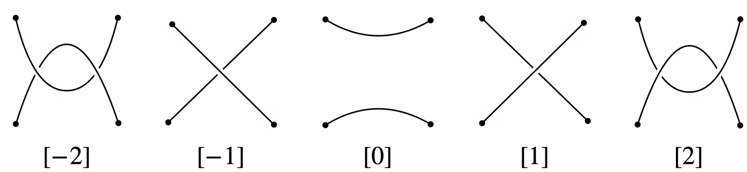

FIGURE 1. Integer tangles.

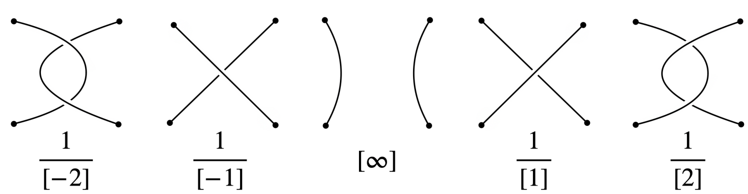

FIGURE 2. Vertical tangles.

If we consider the set $\mathfrak{T}$ to include all rational tangles, then the operations T and R are functions from $\mathfrak{T} \to \mathfrak{T}$. We then introduce a group, $(\Gamma, \circ)$, which includes the collection of all such functions under composition.

Theorem 2.3 ([2])**.** *The collection of all finite combinations of T and R forms a group $(\Gamma, \circ)$ under composition. The group has the presentation:*

$$\Gamma = \langle T, R \,|\, R^2 = I = (TR)^3 \rangle.$$

From these properties, we can form a right action of Γ on $\mathbb{Q}^*$ by defining functions t and r such that

$$t(x) = x + 1 \qquad r(x) = -\frac{1}{x}.$$

2.2. **Tangle Arithmetic.** A more systematic way of characterising and manipulating tangles can be derived from T and R by defining additive and multiplicative operations on tangles.

Definition 2.4 ([1]). Let G be a rational tangle with tangle number x. Then,

(1) Making n twists on the right of a tangle is expressed as:
$$G + [n] \rightsquigarrow x + n.$$
(2) Making n twists on the bottom of a tangle is expressed as:
$$G * \frac{1}{[n]} \rightsquigarrow \frac{1}{n + \frac{1}{x}}.$$

We also consider Lemma 2 from Kauffman and Lambropoulou's paper [1].

Lemma 2.5 (KL Flipping Lemma [1]). *We denote T^{hflip} as the tangle obtained from T by a $180°$-rotation around a horizontal axis on the plane of T, and T^{vflip} as the tangle obtained from T by a $180°$-rotation around a vertical axis on the plane of T. If T is rational, then:*
$$(i)\ T \sim T^{hflip} \text{ and } (ii)\ T \sim T^{vflip}.$$

A consequence of this lemma is the commutativity of tangle addition and multiplication, as summarised in the following corollary. Importantly, this implies that all rational tangles can be constructed using only bottom and right twists, as top and left twists can be flipped to the other side with no effects on the tangle number, as shown.

Corollary 2.6 ([1]). *For $m, n \in \mathbb{Z}$ and rational tangle T:*
$$[m] + T + [n] \sim T + [m+n],\ \frac{1}{[m]} * T * \frac{1}{[n]} \sim T * \frac{1}{[m+n]}.$$

This is key to tangle arithmetic: it allows us to assign every rational tangle a tangle number.

Theorem 2.7 (Continued Fraction Theorem [1]). *Let G be a rational tangle and x be its tangle number. Then G is equivalent to a tangle with a continued fraction representation, where $a_1, \ldots, a_n \in \mathbb{Z}^+ \cup 0$ or $\mathbb{Z}^- \cup 0$, and n is an odd positive integer:*
$$G = [[a_1], [a_2], [a_3], \ldots, [a_n]] = [a_1] + \cfrac{1}{[a_2] + \cfrac{1}{[a_3] + \cdots + \cfrac{1}{\ddots + \cfrac{1}{[a_n]}}}},$$
where its tangle number can be expressed as:
$$x = [a_1, a_2, a_3, \ldots, a_n] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}.$$

3. From Rational Tangles to Rational Numbers

The faithful representation of the tangle group using rational numbers and infinity ($\mathbb{Q}^*$) allows us to establish a relationship between the mapping of rational numbers and geometric transformations on rational tangles. We begin by drawing algebraic parallels between the two.

3.1. **A Geometric Interpretation.** As established, all rational tangles can be constructed using a series of right and bottom twists. Through algebraic manipulation, we express the inverses of the right and bottom twist operations in terms of T_1, T_2, and their inverses.

Lemma 3.1 (The Inverse Right Twist Lemma)**.** $T_2 \circ T_1^{-1}(x) = t^{-1}(x) = x - 1$.

Proof. The functional representation of a right twist is defined as $t(x) = x + 1$. Because this is an injective function, there exists an inverse $t^{-1}(x)$ such that $t^{-1} \circ t(x) = x$. Since

$$T_2 \circ T_1^{-1}(x) = T_2(\frac{1}{2-x}) = x - 1,$$

substituting $x = t(x)$ yields

$$T_2 \circ T_1^{-1}(t(x)) = t(x) - 1 = x.$$

□

Lemma 3.2 (The Inverse Bottom Twist Lemma)**.** $T_2^{-1} \circ T_1(x) = b^{-1}(x) = \frac{1}{-1+\frac{1}{x}}$

Proof. The addition of bottom twists to a rational tangle T with tangle number x is defined as

$$T * \frac{1}{[n]} \rightsquigarrow \frac{1}{n + \frac{1}{x}}.$$

Letting $n = 1$, we form a functional representation of the addition of 1 bottom twist:

$$b : \mathbb{Q}^* \to \mathbb{Q}^*, b(x) = \frac{1}{1 + \frac{1}{x}}.$$

Since this is an injective function, there exists an inverse $b^{-1}(x)$ such that $b^{-1} \circ b(x) = x$. We consider the following:

$$T_2^{-1} \circ T_1(x) = T_2^{-1}(2 - \frac{1}{x}) = \frac{1}{-1 + \frac{1}{x}}.$$

Substituting $x = b(x)$ yields

$$T_2^{-1} \circ T_1(b(x)) = \frac{1}{-1 + \frac{1}{b(x)}} = x.$$

□

Since every rational number corresponds to a class of rational tangles with the same tangle number, and every rational number can be written in the form of a continued fraction, the reduction of rational numbers is a faithful representation of the untangling process of a rational tangle, where the operations introduced previously can be used to undo positive right and bottom twists.

3.2. **Reduction of Rational Numbers.** The invertibility of functions warrants a method of untangling a tangle through the inverses of b and t. To determine this, we first present a method of expressing a rational number as a continued fraction.

Proposition 3.3 (Continued Fraction Form of Positive Rational Numbers)**.** Every positive rational number can be expressed as a continued fraction through the Euclidean algorithm.

Proof. Consider the positive rational number $\frac{a}{b}, a \perp b, a, b \in \mathbb{Z}^+$. We can apply the Euclidean algorithm as follows:

$$\begin{aligned} a &= q_1 b + r_1 & \frac{a}{b} &= q_1 + \frac{r_1}{b} \\ b &= q_2 r_1 + r_2 & \frac{b}{r_1} &= q_2 + \frac{r_2}{r_1} \\ &\vdots & &\vdots \\ r_{n-3} &= q_{n-1} r_{n-2} + r_{n-1} & \frac{r_{n-3}}{r_{n-2}} &= q_{n-1} + \frac{r_{n-1}}{r_{n-2}} \\ r_{n-2} &= q_n r_{n-1} & \frac{r_{n-2}}{r_{n-1}} &= q_n, \end{aligned}$$

where $q_1 \in \mathbb{Z}^+ \cup \{0\},\ n, r_1, \ldots, r_{n-1}, q_2, \ldots, q_n \in \mathbb{Z}^+, b > r_1 > r_2 > \cdots > r_{n-1}$. We note that $q_n \geq 2$ ($r_{n-2} > r_{n-1}$ and $r_{n-1} \mid r_{n-2}$) and that the algorithm is a finite process (the terms are decreasing). As such, we can express $\frac{a}{b}$ as a continued fraction with finite elements:

$$\frac{a}{b} = q_1 + \frac{1}{\frac{b}{r_1}} = q_1 + \cfrac{1}{q_2 + \cfrac{r_1}{r_2}} = q_1 + \cfrac{1}{q_2 + \cfrac{1}{\ddots + \cfrac{1}{q_n}}} = [q_1, q_2, \ldots, q_n]$$

□

This systematic method of converting rational numbers gives rise to our method.

Theorem 3.4 (Rational Reduction Theorem)**.** *All rational numbers can be reduced to 0 using T_1, T_2, and their inverses.*

Proof. For negative rationals, we first map it to a positive rational number using T_2. Consider the positive rational number $\frac{a}{b}$, which can be expressed as

$$\frac{a}{b} = [q_1, q_2, \ldots, q_n] \rightsquigarrow [[q_1], [q_2], \ldots, [q_n]] = ([q_n] * \frac{1}{[q_{n-1}]} + [q_{n-2}]) * \cdots + [q_1].$$

(If n is even, the expression is changed to $[q_1, q_2, \ldots, q_{n-1}, 1]$ to satisfy the canonical form of tangles.) Knowing the construction of the tangle that corresponds to the number enables us to use the inverse functions of right and bottom twist to untwist the tangle to the [0] tangle. The reduction sequence can be expressed as

$$(T_2 \circ T_1^{-1})^{q_n} \circ \cdots \circ (T_2^{-1} \circ T_1)^{q_2} \circ (T_2 \circ T_1^{-1})^{q_1}(x).$$

□

We illustrate this in the following example.

Example 3.5. Consider the rational number $\frac{7}{5}$.

$$\begin{aligned} 7 &= 1 \times 5 + 2 \\ 5 &= 2 \times 2 + 1 \\ 2 &= 2 \times 1 + 0 \end{aligned}$$

Therefore $\frac{7}{5} = [1, 2, 2]$. The reduction sequence yields:

$$\begin{aligned}(T_2 \circ T_1^{-1})^2 \circ (T_2^{-1} \circ T_1)^2 \circ (T_2 \circ T_1^{-1})(\frac{7}{5}) &= (T_2 \circ T_1^{-1})^2 \circ (T_2^{-1} \circ T_1)^2(\frac{2}{5}) \\ &= (T_2 \circ T_1^{-1})^2(2) \\ &= T_2 \circ T_1^{-1} \circ T_2(\infty) \\ &= T_2 \circ T_1^{-1}(1) \\ &= 0\end{aligned}$$

3.3. **Construction of Rational Numbers.** Mirroring the inverse right and bottom twist operations, we express the right and bottom twist functions in terms of T_1, T_2, and their inverses.

Lemma 3.6 (The Right Twist Lemma)**.** $T_1 \circ T_2^{-1}(x) = t(x) = x + 1$

Proof. $T_1 \circ T_2^{-1}(x) = T_1(\frac{1}{1-x}) = x + 1 = t(x)$ □

Lemma 3.7 (The Bottom Twist Lemma)**.** $T_1^{-1} \circ T_2(x) = b(x) = \frac{1}{1+\frac{1}{x}}$

Proof. $T_1^{-1} \circ T_2(x) = T_1^{-1}(1 - \frac{1}{x}) = \frac{1}{1+\frac{1}{x}} = b(x)$

□

As such, we can generalise the problem to include the construction of any rational number from any other rational number.

Corollary 3.8 (Generalised Rational Reduction)**.** *Any rational number can be mapped to any other rational number through T_1, T_2, and their inverses.*

Proof. Suppose we would like to map the rational number $x = [q_1, \ldots, q_n]$ to $y = [p_1, \ldots, p_m]$. We first reduce the rational using the sequence

$$(T_2 \circ T_1^{-1})^{q_n} \circ \cdots \circ (T_2^{-1} \circ T_1)^{q_2} \circ (T_2 \circ T_1^{-1})^{q_1}(x).$$

Then, we can construct the rational y using the right and bottom twist operations:

$$(T_1 \circ T_2^{-1})^{p_m} \circ \cdots \circ (T_1^{-1} \circ T_2)^{p_2} \circ (T_1 \circ T_2^{-1})^{p_1}(0) = y.$$

□

Example 3.9. Suppose we would like to map $\frac{3}{8}$ to $\frac{5}{2}$. We first express them in continued fraction form.

$$\begin{aligned} 3 &= 0 \times 8 + 3 & 5 &= 1 \times 3 + 2 \\ 8 &= 2 \times 3 + 2 & 3 &= 1 \times 2 + 1 \\ 3 &= 1 \times 2 + 1 & 2 &= 2 \times 1 + 0 \\ 2 &= 2 \times 1 + 0 & \frac{5}{2} &= [1, 1, 2] \\ \frac{3}{8} &= [0, 2, 1, 2] = [0, 2, 1, 1, 1] \end{aligned}$$

This yields the sequence

$$(T_1 \circ T_2^{-1})^2 \circ (T_1^{-1} \circ T_2) \circ (T_1 \circ T_2^{-1}) \circ$$
$$(T_2 \circ T_1^{-1}) \circ (T_2^{-1} \circ T_1) \circ (T_2 \circ T_1^{-1}) \circ (T_2^{-1} \circ T_1)^2 \circ (T_2 \circ T_1^{-1})^0(\frac{3}{8}) = \frac{5}{2}.$$

4. Final Remarks

The faithful representation of tangles through the functions introduced in the problem enables the geometric interpretation of rational numbers as rational tangles constructed from a series of right and bottom twists. Such an approach provides physical meaning to the domain of the functions – the inclusion of infinity is not a "mathematical hack"; rather, it refers to the well-defined infinity tangle. The mappings, $\frac{1}{\infty} \mapsto 0, \frac{1}{0} \mapsto \infty$, stem from this: the rotation operation ($r(x) = -\frac{1}{x}$) transforms the [0] and [∞] tangles into each other, which yields the mappings in their respective tangle numbers.

A useful generalisation of the problem is to explore pairs of functions $T_1(x) = m - \frac{1}{x}$ and $T_2(x) = k - \frac{1}{x}$ for $m, k \in \mathbb{Z}$ to investigate the conditions for which a general method of reducing fractions exists and whether they relate to Conway's rational tangles.

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References

[1] Kauffman, L & Lombropoulou, S 2004, On the classification of rational tangles, pdf, viewed 11 July 2020, <https://arxiv.org/pdf/math/0311499.pdf>.

[2] Prof. Salomone, M 2018, OAK Rational Tangles, Open Algebra and Knots, viewed 3 June 2020, <http://matthematics.com/oak/chapter-tangles.html>.

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