Fluid approach to total-progeny-dependent branching processes

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Introduction

A continuous-time total-progeny-dependent branching process is a 2-dimentional Markov chain, (Z_t, X_t), where Z_t is the population size at time t and X_t is the total progeny until time t ($t \in \mathbb{R}^+$).

- $b(x)$ and $d(x)$: individual birth rate and death rate depending on the current total progeny x ($x \in \mathbb{N}$).
- Transition probabilities:
  - $(z, x) \to (z+1, x+1)$ w.p. $\frac{b(x)}{b(x)+d(x)}$,
  - $(z, x) \to (z-1, x)$ w.p. $\frac{d(x)}{b(x)+d(x)}$.

We consider two simple models with death rate $d(x) = \mu$ and birth rate:

- **Model 1**: $b_1(x) = \frac{\lambda}{x}$
- **Model 2**: $b_2(x) = \lambda e^{-\alpha x}$,

where λ, α, μ are all constant parameters. For both models, extinction happens with probability 1.

Objectives

- Study the stochastic process from its *fluid approximation*
- **Quantities of interest:**
  - the maximum population size
  - the total progeny at extinction
  - the extinction time

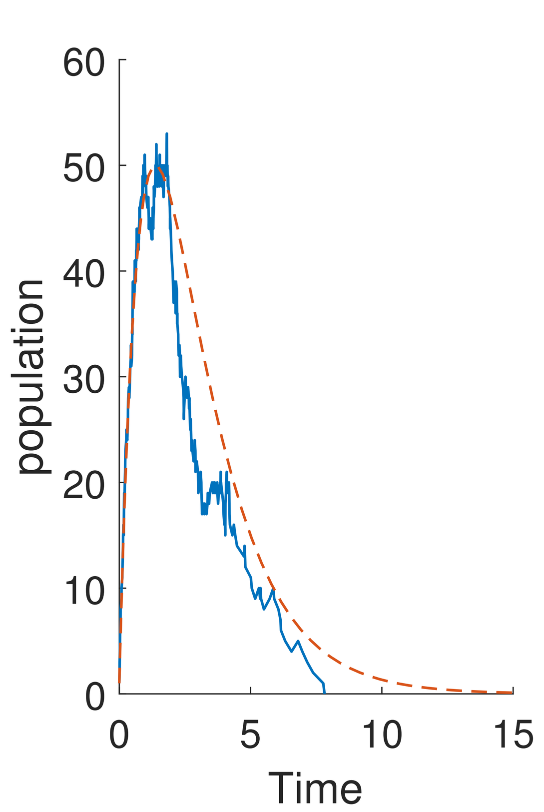

Figure 1: Model 1: Single trajectory, $\mu = 1, \lambda = 100$.

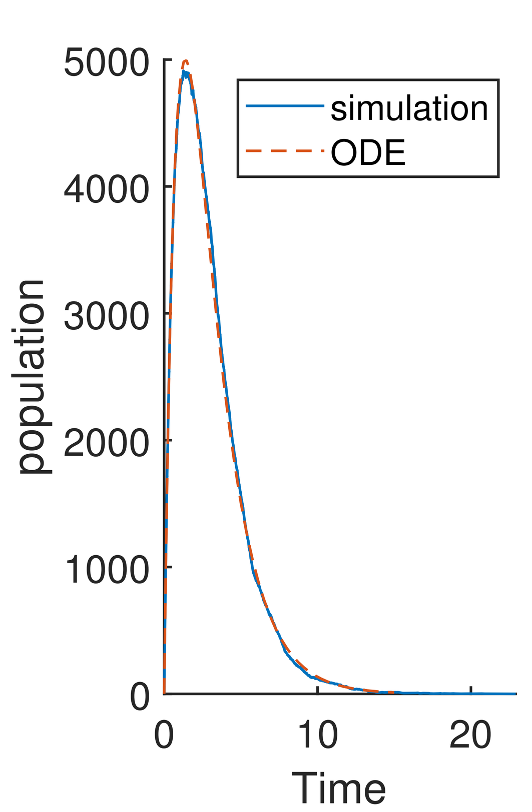

Figure 2: Model 1: Single trajectory, $\mu = 1, \lambda = 10000$.

Associated fluid approximation

From Darling [2], it is known that the stochastic process (Z_t, X_t) has a fluid (deterministic) approximation $(y_1(t), y_2(t)), t \geq 0$, which satisfies:

$$\frac{dy_1}{dt} = y_1(b(y_2) - d(y_2)) \quad (1)$$

$$\frac{dy_2}{dt} = y_1 b(y_2). \quad (2)$$

In our models, we set the initial conditions to be:

$$y_1(0) = y_2(0) = 1. \quad (3)$$

Single simulation trajectories and associated ODE approximations are shown in Figures 1 and 2.

Solving ODEs

In general, by dividing Equation (1) by Equation (2), we obtain an expression for y_1 in terms of y_2:

$$\frac{dy_1}{dy_2} = \frac{b(y_2) - d(y_2)}{b(y_2)} \Longrightarrow y_1 = y_2 - \int \frac{d(y_2)}{b(y_2)} dy_2. \quad (4)$$

Therefore, the expressions for y_1 in terms of y_2 are:

- Model 1: $y_1 = y_2 - \frac{\mu}{2\lambda} y_2^2 + \frac{\mu}{2\lambda}$
- Model 2: $y_1 = y_2 - \frac{\mu}{\alpha\lambda} e^{\alpha y_2} + \frac{\mu}{\alpha\lambda} e^{\alpha}$

Results from ODEs

- Model 1: *Explicit expressions* are attainable for all quanities of interest in terms of the parameters
- Model 2: Only implicit expressions are available, but can be used for numerical analysis
- Solutions from ODEs *give good approximations for quantities of interest except the extinction time*

Estimating the mean extinction time

As y_1 in the ODEs never reaches 0, we approximate the mean extinction time by the time at which $y_1 = \varepsilon$, with $\varepsilon = 1$. We use two approaches to construct the mean extinction time:

- Calculate t^*, the time when the total progeny is one less compared to at extinction, i.e. such that $y_2(t^*) = y_2(\infty) - 1$, and construct: $t_{ext}^{(1)} \approx t^* + \mathbb{E}(\max(X_1, \ldots, X_{y_1(t^*)}))$, where $X_i \sim \exp(\mu)$.
- Assume the results from the ODEs approximate the last occurrence of $y_1 = 2$ and its corresponding time $t_{\varepsilon=2}$ well. Construct: $t_{ext}^{(2)} \approx t_{\varepsilon=2} + \mathbb{E}(\max(X_1, X_2))$.

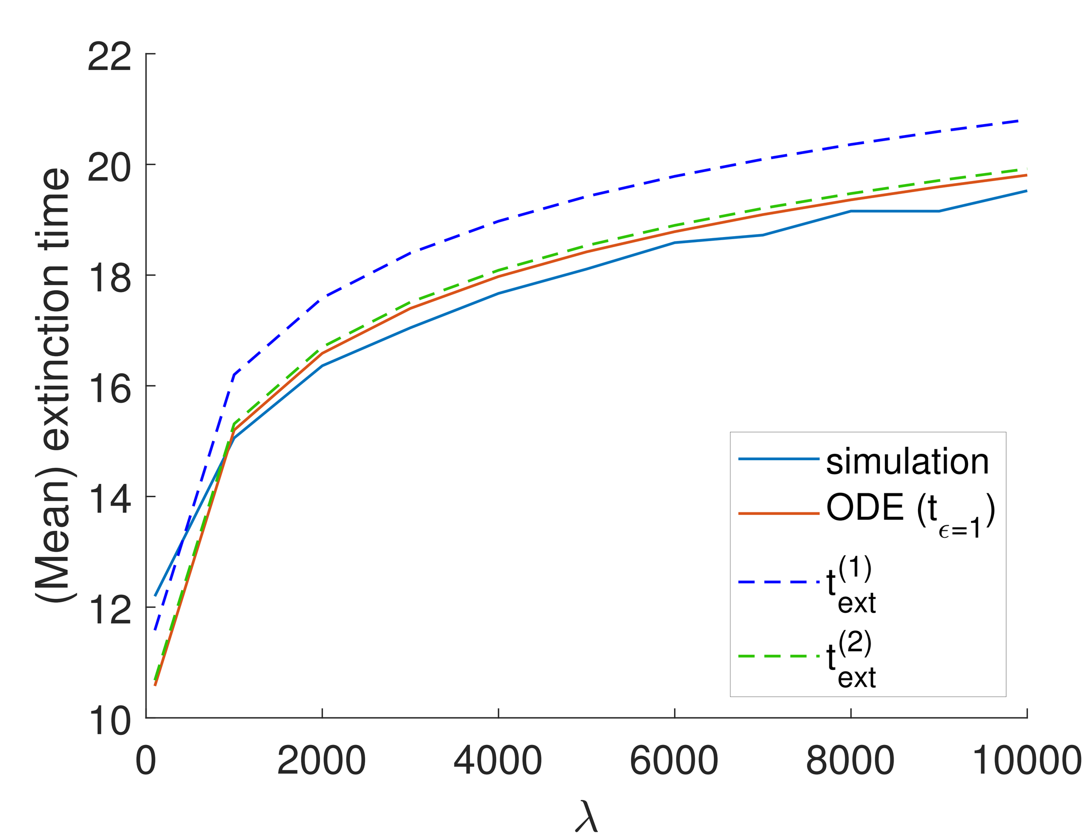

Figure 3: Model 1 extinction time vs λ, $\mu = 1$.

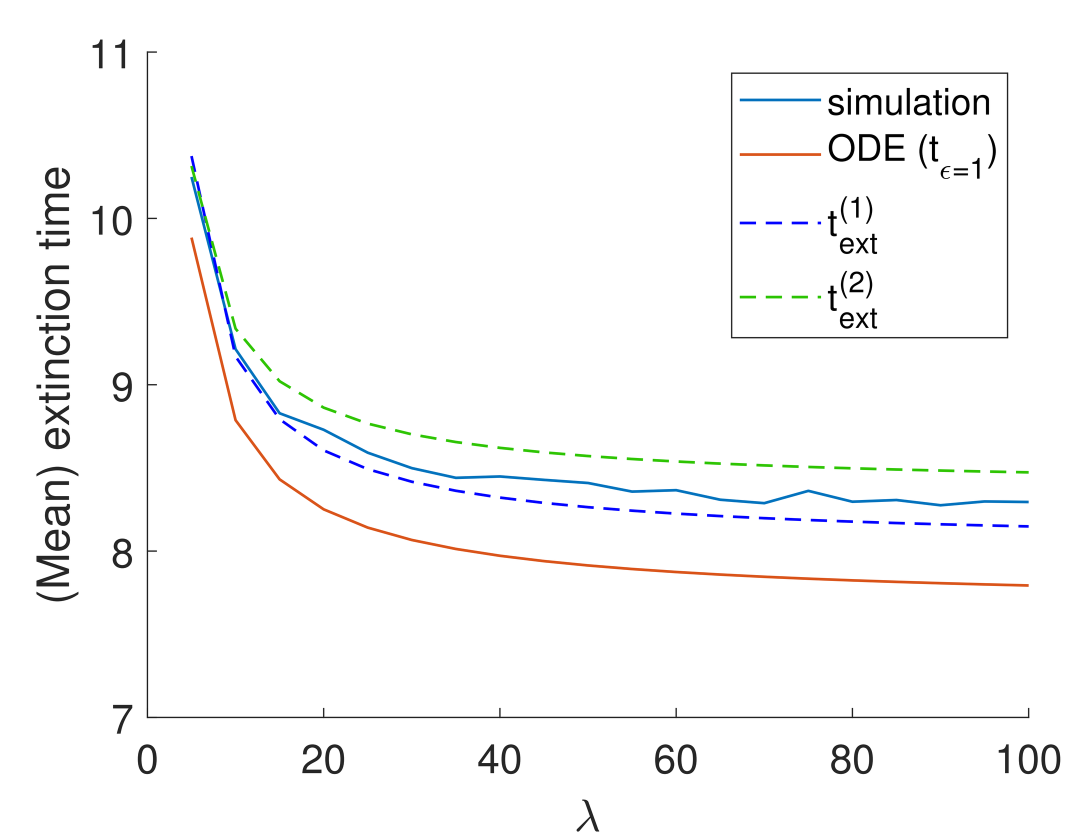

Figure 4: Model 2 extinction time vs λ, $\alpha = 0.01, \mu = 1$.

The mean extinction time vs the parameters

Model 1: From Figure 3, we can see that the mean extinction time increases with λ.

Model 2: We can make two observations:

- Holding α, μ constant, the mean extinction time first decreases (Figure 4 shows the phase of decrease) and then increases as we increase λ,
- Holding λ, μ constant, the mean extinction time decreases as we increase α.

It is interesting to note that *increasing λ and α has opposite effects* on the individual birth rate $b(x)$, but for small values of λ, an increase in λ and α both reduce the mean extinction time.

Work in progress

As mentioned in the previous section, there appears to be a *minimum* for the mean extinction time t_{ext} as a function λ in Model 2. It is interesting to investigate:

- What value of λ would attain the minimum for t_{ext},
- How t_{ext} would behave as $\lambda \to \infty$.

The experience

The vacation research scholarship has offered me the invaluable opportunity to gain great insights from the research side of mathematics. I would like to say a special thank you to my supervisors, Dr. Sophie Hautphenne and Dr. Brendan Patch, for the constant help and the passionate guidance that allowed me to have a very pleasant learning experience.

References

[1] Linda JS Allen. An introduction to stochastic epidemic models. In *Mathematical epidemiology*, pages 81–130. Springer, 2008.

[2] RWR Darling. Fluid limits of pure jump markov processes: a practical guide. *arXiv preprint math/0210109*, 2002.