Character theory of finite groups

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Background

The representation theory of groups is the study of representing groups as vector space automorphisms or matrices. That is, a representation of a group on a vector space V is a group homomorphism $\rho : G \to GL(V)$. We say V is a G-module as it has a (left) G-linear action $gv \to \rho(g)v$. A G-module is irreducible if no proper subspace of it is a G-module ([1]). An example of such a module is the trivial G-module with representation:

$$\rho : G \to GL_1(\mathbb{C}), \rho(g) = 1$$

We consider the case where G is a finite group and the underlying field is complex. Then we can show that there exist a finite number of irreducible G-modules $V_1, V_2, \ldots, V_k$ such that every irreducible G-module is isomorphic to one of them. In addition, every G-module V is isomorphic to a direct sum by Maschke's theorem:

$$V \cong d_1 V_1 \oplus d_2 V_2 \oplus \cdots \oplus d_k V_k$$

where $d_1, d_2, \ldots, d_n$ are non-negative integers representing the multiplicities of $V_1, V_2, \ldots, V_k$.

Burnside's Theorem

Character theory provides us another perspective to think about group theoretic problems. For example, Burnside's theorem is hard to prove purely with group theory, but it can be proved (relatively) easily with characters. It states that every group of order $p^a q^b$ is not simple, where p, q are primes and a, b are non-negative integers with $a + b \geq 2$ ([2]). Furthermore, such groups are solvable, which means that there is a chain of subgroups of G

$$\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G$$

where for all i, G_i is a normal subgroup of G_{i+1} and G_{i+1}/G_i is cyclic with prime orders.

References

[1] William Fulton and Joe Harris. *Representation theory : a first course*. Graduate texts in mathematics Readings in mathematics: 129. New York: Springer-Verlag, 1991. ISBN: 0387974954.

[2] G. D. James and M. W. Liebeck. *Representations and characters of groups*. London: Cambridge University Press, 2001. ISBN: 0521812054.

An example

Consider $G = D_6 = \langle r, s \mid s^2 = r^6 = e, rs = sr^{-1} \rangle$, the dihedral group of order 12. The symmetries of a hexagon can be characterised by the action of G on its vertices. As illustrated in Figure 1, r is a rotation and s is a reflection on the hexagon.

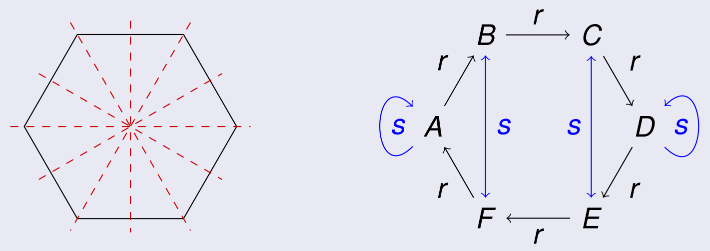

Figure 1: The symmetries of a hexagon (left) and how r and s acts on it (right).

The character table of G is shown in Table 1. Here, χ_1 is the trivial character and $C_G(g)$ is the centraliser of g. Notice that $\chi_2(r^2) = \chi(e) = 1$, so its kernel $\langle r \rangle$ is a normal subgroup of G and hence G non-simple.

g	e	r^2	r	r^3	s	sr
$\lvert C_G(g) \rvert$	12	4	4	12	6	6
χ_1	1	1	1	1	1	1
χ_2	1	1	1	1	−1	−1
χ_3	1	1	−1	−1	1	−1
χ_4	1	1	−1	−1	−1	1
χ_5	2	−1	1	−2	0	0
χ_6	2	−1	−1	2	0	0

Table 1: Character table of $G = D_6$

Now this group has order $12 = 2^2 \times 3^1$, so it should be solvable by Burnside's theorem (left). Indeed,

$$\{e\} \trianglelefteq \langle r^3 \rangle \trianglelefteq \langle r \rangle \trianglelefteq G$$

$$G/\langle r \rangle \cong C_2 \quad \langle r \rangle / \langle r^3 \rangle \cong C_3 \quad \langle r^3 \rangle / \{e\} \cong C_2$$

So all G_{i+1}/G_i are cyclic with prime orders.

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Characters

The character function of a representation is simply the trace of the automorphism in V. We say the character is irreducible if the corresponding G-module is irreducible. Here are some useful properties of characters:

- $\chi(e)$ is equal to the dimension of the corresponding G-module.
- Characters are class functions, so $\chi(g) = \chi(h)$ if g and h are conjugate.
- For any $g \in G$, let $n = \chi(e)$, $m = ord(g)$. Then $\chi(g) = \omega_1 + \cdots + \omega_n$, where ω_i is an m-th root of unity. Hence their values are always algebraic integers.
- $\chi(g) = \overline{\chi(g^{-1})}$ as the underlying field is $\mathbb{C}$.
- The number of distinct irreducible characters is equal to the number of conjugacy classes of G.
- Let U, V be G-modules. If χ is the character of U and ψ the character of V, then $\chi + \psi$ is the character of $U \oplus V$ and $\chi\psi$ is the character of $U \otimes V$.

As a consequence of the complete reducibility of G-modules, there exists irreducible characters $\chi_1, \chi_2, \ldots, \chi_k$ such that every character has the form:

$$\chi = d_1\chi_1 + d_2\chi_2 + \cdots + d_k\chi_k$$

where $d_1, d_2, \ldots, d_k$ are non-negative integers.
We can also define an inner product on the characters. Let $[g_1], [g_2], \ldots, [g_k]$ be all the distinct conjugacy classes of G. Then we have:

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi(g)} = \sum_{i=1}^{k} \frac{\chi(g_i)\overline{\psi(g_i)}}{|C_G(g_i)|}$$

The second formula uses the fact that characters are class functions, so they have the same value over any given conjugacy class of G. The irreducible characters are orthonormal with respect to this inner product. Therefore, by orthonormal projection, the multiplicities of the irreducible characters can be computed easily with $d_i = \langle \chi, \chi_i \rangle$.
The irreducible characters are usually presented in a table, which displays the following properties:

- The rows correspond to the distinct irreducible characters, while the columns represent the distinct conjugacy classes of G.
- The number of rows is equal to the number of columns, as the number of irreducible characters is equal to the number of conjugacy classes.
- The rows (and columns) of the table are orthogonal, as a result shown above: $\langle \chi_i, \chi_j \rangle = \delta_{i,j}$