An exploration into Quantum Groups and their application in Quantum Mechanics.

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Introduction

Quantum groups were first introduced in the 1980s by Drinfield and Jimbo, as a way to solve the Yang-Baxter equation. They are noncommutative algebras that can be deformed. They were independently constructed from a mathematical physics perspective, and one of statistical physics. This project explored three different quantum groups, and worked to identify their application in quantum mechanics.

FIGURE 1: AN ARTISTS IMPRESSION OF QUANTUM PHYSICS

Methodology:

$$U_q \rightarrow R \rightarrow M \rightarrow T \rightarrow H$$

FIGURE 2:DIAGRAM DEPICTING THE LINK BETWEEN QUANTUM GROUPS AND QUANTUM MECHANICS

The universal R matrix is a solution to the Yang-Baxter equation derived from a quantum group. A monodromy matrix is derived from the R matrix, then the transfer matrix. Finally, the Hamiltonian for a quantum system is formed.

U_q

The generators and commutation relations for each group are shown below:

- The Heisenberg algebra:

$$\{a_n, a_{-n}\}_{n>0}$$
$$[a_n, a_{-m}] = \delta_{n,m} n$$

- Quantum sl_2:

$$e, f, k = q^h$$
$$kek^{-1} = q^2 e, \quad kfk^{-1} = q^{-2} f, \quad [e,f] = \frac{k-k^{-1}}{q-q^{-1}}$$

- Quantum affine sl_2:

$$e_0, f_0, e_1, f_1, k_0, k_1$$
$$q^x e_i q^{-x} = q^{\alpha_i(x)} e_i, \quad q^x f_i q^{-x} = q^{-\alpha_i(x)} f_i$$
$$[e_i, f_j] = \frac{1}{q-q^{-1}} \delta_{ij} (q^{h_i} - q^{-h_i})$$

R

The calculation of the R matrix is through the use of a Drinfield double, where a_i are basis elements.

$$R = \sum_i^{\infty} a_i \otimes a_i^* \qquad \begin{matrix} R \in A \hat{\otimes} A^\circ \\ a_i \in A \\ a_i^* \in A^\circ \end{matrix}$$

The matrix obtained in this fashion obeys the following relations:

$$R\Delta(a) = \Delta^{op}(a) R$$
$$(1 \otimes \Delta) R = R_{13} R_{23}, \ (\Delta \otimes 1) R = R_{13} R_{23}$$

These properties alone are enough to prove that R is a solution to the Yang-Baxter Equation.

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

$M, T \ \& \ H$

The following relations depict how one can obtain the Hamiltonian from the universal R matrix for periodic boundary conditions. When T has been diagonalised, so has H. This is incredibly useful when it is difficult to diagonalise H directly.

$$M_N(w) = \Pi_{k=1}^N R_{0k}\left(\frac{z_k}{w}\right)$$
$$T(w) = \mathrm{Tr}_0(M_N(w))$$
$$H := T^{-1}(1) \left.\frac{\partial}{\partial w} T(w)\right|_{w=1}$$

Results:

$$R = \begin{bmatrix} qz - q^{-1}z^{-1} & 0 & 0 & 0 \\ 0 & z - z^{-1} & (q-q^{-1})z & 0 \\ 0 & (q-q^{-1})z^{-1} & z - z^{-1} & 0 \\ 0 & 0 & 0 & qz - q^{-1}z^{-1} \end{bmatrix}$$

FIGURE 3: R, A REPRESENTATION OF THE UNIVERSAL R MATRIX OF $U_q(\mathfrak{sl}_2)$ WITH z THE SPECTRAL PARAMETER.

$$H = -\frac{2}{q-q^{-1}} \begin{bmatrix} q + q^{-1} + 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2q & 0 \\ 0 & 0 & 0 & q + q^{-1} + 1 \end{bmatrix}$$

FIGURE 4: H, A HEISENBERG MATRIX CORRESPONDING TO M_2

Discussion:

The final Hamiltonian matrix found in *figure 4* is a great example of how a method like this can work. The use of these theoretical algebra structures to learn information about physical quantum systems is incredibly fascinating and useful. The three groups studied in the scope of this project each had their unique properties, yet there are many more quantum groups out there with links to other physical systems. A more comprehensive review would explore these. There is always room for improvement: *figure 4* has yet to be diagonalised. Hence, further research into techniques of diagonalisation, and computational methods to find the Heisenberg matrix using the monodromy matrix with a general N would be informative.

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References: eigenchris. (2018, March 3rd.) *Tensors for Beginners: Tensor Product Space*[Video]. YouTube. https://www.youtube.com/watch?v=M-OLmxuLdbU. Radford, D. E. (1993). Minimal Quasitriangular Hopf Algebras. *Journal of Algebra, 157(2),* 285-315. https://doi.org/10.1006/jabr.1993.1102. Faddeev, L.D. (2006). History and Perspectives of Quantum Groups. *Milan j. math. 74,* 279–294. https://doi.org/10.1007/s00032-006-0062-4. Wikipedia. (2022, April 12th). *Quantum Group*. https://en.wikipedia.org/wiki/Quantum_group. Negut, A. (2020). The R-matrix of the quantum totoidal algebra. *arxiv*. https://arxiv.org/abs/2005.14182. Kharchev, S., Pakuliak, S. Lectures on Quantum Affine Algebras. Boos, H., Göhmann, A., Nirov, Kh. S., Razumov, A. V. (2012). Universal Integrability Objects. *arxiv*. https://doi.org/10.1007/s11232-013-0002-8