# Diagram groups

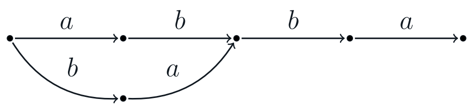
Leigh Greville

THE UNIVERSITY OF MELBOURNE

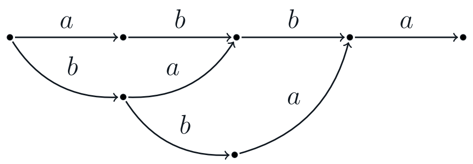
## Diagrams

Let $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$ be a monoid presentation. A **diagram** over $\mathcal{P}$ is a representation of a derivation in $\mathcal{P}$ as a graph. Let $\mathcal{P} = \langle a, b \mid ab = ba \rangle$. We will construct the diagram corresponding to the derivation $(ab)ba \to (ba)ba = b(ab)a \to bbaa$. We begin with the **elementary diagram** $\epsilon(abba)$:


We then attach the **cell** corresponding to the rewrite $ab \to ba$ (written $\Psi_{ab,ba}$):




This is an **atomic diagram** as it consists of exactly one cell. Finally, we attach another $\Psi_{ab,ba}$ cell:




The **top path** of this diagram $(\mathrm{top}(\cdot))$ is $abba$ and the **bottom path** $(\mathrm{bot}(\cdot))$ is $bbaa$. We may **concatenate** two diagrams $D_1, D_2$ if $\mathrm{bot}(D_1) = \mathrm{top}(D_2)$ by identifying $D_1$'s bottom path and $D_2$'s top path. This concatenation is denoted $D_1 \circ D_2$. We may **add** any two diagrams $D_1, D_2$ by identifying $D_1$'s rightmost vertex with $D_2$'s leftmost. The resulting diagram is denoted $D_1 + D_2$.

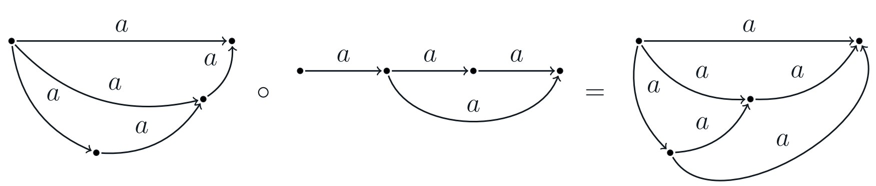


Fig. 1: Concatenation

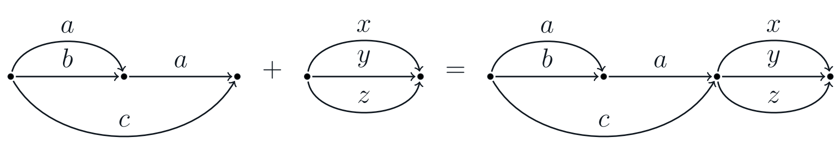
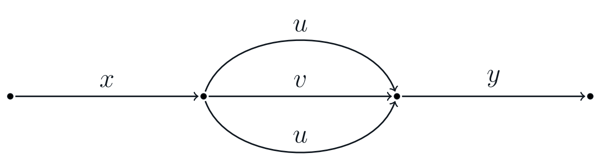


Fig. 2: Addition

## Dipoles and diagram groups

A **dipole** is a configuration of cells in a diagram that looks like this:




Where $x, y, u, v \in \Sigma^*$.

More formally, a diagram $D$ contains a dipole if it can be written as the composition: $P \circ A \circ A^{-1} \circ S$, where $A = \epsilon(x) + \Psi_{u,v} + \epsilon(y)$ and $A^{-1} := \epsilon(x) + \Psi_{v,u} + \epsilon(y)$. In terms of derivations, a dipole corresponds to applying a rewrite rule and then immediately undoing it. So, we will remove dipoles where we find them. Repeatedly removing dipoles, in any order, until there are none left, will always produce the same **reduced** diagram [4]. We will define a new operation on diagrams, $\cdot$, which is concatenation $\circ$ followed by reduction. This gives the set of diagrams a groupoid structure, where the inverse of a diagram is given by applying the inverse rewrites in reverse order. If we fix a word $w \in \Sigma^*$, and only consider diagrams $D$ such that $\mathrm{top}(D) = \mathrm{bot}(D) = w$ (**spherical** diagrams), the resulting substructure is the **diagram group** $D(\mathcal{P}, w)$.

## Squier's complex

**Squier's complex** (denoted $K(\mathcal{P})$ or $S(\mathcal{P}, w)$) is a cube complex whose vertices are the words of $\Sigma^*$ and whose $n$-cubes are given by:

$$(a_1, u_1 \to v_1, ..., a_n, u_n \to v_n, a_{n+1})$$

For $a_i \in \Sigma^*$ and $(u_i = v_i) \in \mathcal{R}$.
E.g, if $(u_1 = v_1), (u_2 = v_2) \in \mathcal{R}$, then for all $x, y, z \in \Sigma^*$ we have the square:

$$\begin{tikzcd}
xu_1yu_2z \arrow[r] \arrow[d] & xv_1yu_2z \arrow[d] \\
xu_1yv_2z \arrow[r] & xv_1yv_2z
\end{tikzcd}$$

The cubes in Squier's complex correspond to independent applications of the rewrite rules in $\mathcal{R}$. A very important result about diagrams is that their groupoid structure is isomorphic to the groupoid of paths in Squier's complex (up to homotopy). In particular, $D(\mathcal{P}, w) \cong \pi_1(K(\mathcal{P}), w)$. Fundamental groups of cube complexes are already well studied, so we now have many results that we can apply to diagram groups.

## Presentations of diagram groups

There is a well-known method for obtaining a presentation of $\pi_1(X, p_0)$, where $X$ is a cube-complex: choose a spanning tree $T$ of the connected component of $X$ containing $p_0$. Then $\pi_1(X, p_0)$ is generated by the edges in this connected component, subject to the relations:

1. $\forall e \in T,\ e = 1$
2. If $e_1, e_2, e_3, e_4$ form the (oriented) boundary of a square in $X$, then $e_1e_2e_3e_4 = 1$

For example, if $\mathcal{P} = \langle a, b, c \mid a = b,\ a = c,\ b = c \rangle$, this result quickly shows that $D(\mathcal{P}, a) \cong \mathbb{Z}$. With a bit more effort, one can show that Thompson's group $F$ is isomorphic to a very simple to state diagram group: $D(\langle a \mid a = a^2 \rangle, a)$.

## Hyperplanes of Squier's complex

The hyperplanes of Squier's complex can also give information about the corresponding diagram group. For example, cutting Squier's complex along a set of 'well-behaved' hyperplanes gives a decomposition of the diagram group as a graph of groups [2, thm. 5.8]. Properties of these hyperplanes are often equivalent to easy to check statements about the monoid presentation $\mathcal{P}$.

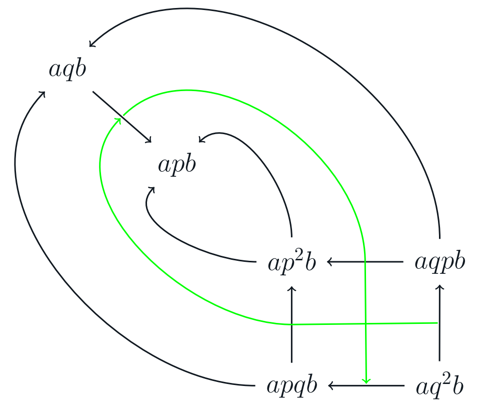



Fig. 3: A self-intersecting hyperplane in $S(\langle a, b, p, q \mid ap = a,\ pb = b,\ q = p \rangle, ab)$

## Some results

- Diagram groups are torsion free [3, thm. 15.11]
- The abelianisation of any diagram group is free abelian [3, thm. 11.2]
- A diagram group is free if and only if it does not contain a subgroup isomorphic to $\mathbb{Z}^2$ [1, thm. 4.1]
- Diagram groups over semigroup presentations have solvable word problem [3, thm. 14.2]

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## References

[1] Anthony Genevois. *Hyperbolic diagram groups are free.* 2015. arXiv: 1505.02053.

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[3] V. Guba and M. Sapir. *Diagram Groups.* American Mathematical Society, 1997.

[4] Vesna Kilibarda. "On the Algebra of Semigroup Diagrams". In: *International Journal of Algebra and Computation* 07.03 (1997), pp. 313–338.