Explorations of the Bethe Ansatz at first-order

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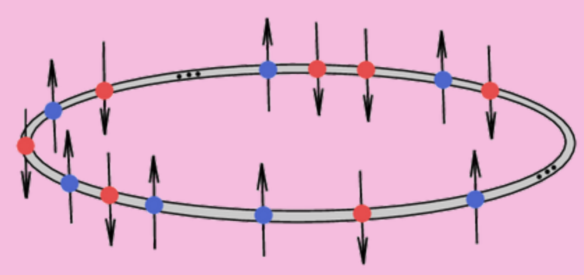
In physical systems, the energy of particles are given by the eigenvalues of the Hamiltonian operator ($\hat{\mathcal{H}}\psi = \Lambda\psi$), with the eigenvectors being physically important states themselves. Consequently, solving this eigenvalue equation is of utmost importance for understanding the behaviour of many systems.

A notable class of problem that appear across condensed matter and particle physics are interacting spins, where lattices or chains of particles interact with each other. Often, it is a reasonable approximation to consider only **"nearest-neighbour" interactions**, situations where only adjacent spins interact. In this project, the focus was upon loops of these type (of length L).



An example of a spin loop (Pribytok, 2022)

Energy eigenvalues

In Fock space, the Hamiltonian takes the block-diagonal form:

$$\hat{\mathcal{H}} = \oint dz \exp\Big(\sum_{n>0} \frac{1-t^{-n}}{n} a_{-n} z^{n}\Big) \exp\Big(-\sum_{n>0} \frac{1-t^{n}}{n} \frac{1-s^{n}t^{n}q^{-n}}{1-s^{n}} a_{n} z^{-n}\Big)$$

Where a_{-n} and a_n are respectively the creation and annihilation operators of the Heisenberg algebra.

The Bethe ansatz

The eigenvalue equation is often very difficult to solve directly. As such, the **Bethe ansatz** is an assumed plane-wave form for the eigenstates:

$$\psi^{A} = \sum_{\substack{\{x_1 \dots x_n\} \\ 1 \le x_1 < \dots x_n \le L}} \Big(\sum_{\sigma \in S_n} A^{\sigma} \prod_{i=1}^{n} z_i^{x_i}\Big) |x_1, \dots, x_n\rangle \qquad \text{(Batchelor, 2007)}$$

Acting on such states with the Hamiltonian produces an expression of the form:

$$\hat{\mathcal{H}}\psi^{A} = \Lambda\psi^{A} + \underbrace{(\dots)}$$

Where, if we require that the remaining terms are zero, the eigenvalue equation is solved. As derived by Feigin, et. al, this produces the **Bethe equations**:

$$(1 - z_i)\prod_{j \neq i} \zeta(z_j/z_i) = s(tq^{-1} - z_i)\prod_{j \neq i} \zeta(z_i/z_j)$$

where $i \in \{1, \dots, L\}$ and $\zeta(x) = \frac{(1-qx)(1-t^{-1}x)}{(1-x)(1-qt^{-1}x)}$

Solved by the **Bethe roots**, z_i. Feigin, et. al. further showed that the eigenvalues are given by:

$$\Lambda = -(1-t^{-1})(1-q)\sum_{i=1}^{L} z_i + 1$$

Order zero solutions

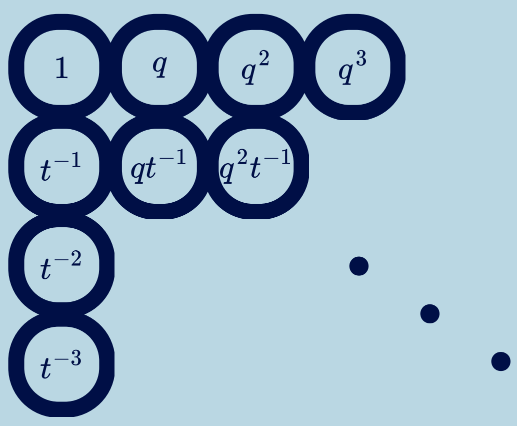
The Bethe equations are non-trivial to solve in general. However, for small s, perform a first-order expansion. What follows is a procedure for solving such equations.

- First, solve the zeroth-order (s=0) Bethe equations:

$$(1 - z_i^{(0)})\prod_{j \neq i} \zeta(z_j^{(0)}/z_i^{(0)}) = 0$$

- The solutions correspond to every integer partition of L and can be elegantly represented on Young diagrams, read off left-right, top-bottom.

$z_i^{(0)}$:



First-order expansion

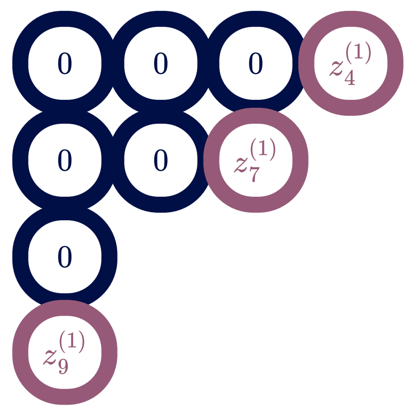
Armed with the zeroth-order solutions, the first order can now be solved.

- To prevent (removable) singularities, it is quintessential to solve in left-right, top-bottom order across the diagram.
- When substituting in the zeroth-order solutions, the RHS vanishes whenever i corresponds to a box other than a bottom-right 'corner'.

$$(1 - z_i)\prod_{j \neq i} \zeta(z_j/z_i) = s(tq^{-1} - z_i^{(0)})\prod_{j \neq i} \frac{\overbrace{(z_j^{(0)} - qz_i^{(0)})}^{\text{when j is a box to the left } 0}\overbrace{(z_j^{(0)} - t^{-1}z_i^{(0)})}^{0 \text{ when j is a box above}}}{(z_j^{(0)} - z_i^{(0)})(z_j^{(0)} - qt^{-1}z_i^{(0)})}$$

- Then solved iteratively, these produce trivial solutions.

$z_i^{(1)}$:



- The full equation (1) must be solved for the corners, which can be done using programs such as Mathematica.
- The crux of the argument is that one of the terms in the product on the left takes the form $s(\dots)z_i^{(1)}$, restricting all other terms to be order s^0.
- Then it remains to solve by simple rearrangement.

Conclusion

Given the zeroth- and first-order Bethe roots, the eigenvalues can be computed:

$$\Lambda^{(0)} = -(1-t^{-1})(1-q)\sum_{(a,b)} q^{a-1}t^{1-b} + 1 \quad \text{where (a,b) label the boxes}$$

$$\Lambda^{(1)} = -(1-t^{-1})(1-q)\sum_{\text{corner boxes, } i} z_i^{(1)}$$

$$\Lambda = \Lambda^{(0)} + s\Lambda^{(1)}$$

Acknowledgements

I would like to extend my sincere gratitude to my supervisor Dr. Sasha Garbali for his patience, support and many office hours explaining theory. To the staff directing the Vacation Scholarship program, your tireless effort to make a fantastic experience does not go unnoticed. Thank you.

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