Constructing minimum Euclidean skeletons for polygons with holes

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How not to crash into walls

Let us model obstacles as polygons. When checking whether our line of motion intersects with an obstacle, we can represent the polygon by its minimum Euclidean skeleton to speed up the intersection test.

Euclidean Skeleton

Let Ω be a polygon in the Euclidean plane that may contain holes and is not self-intersecting. Denote the boundary of Ω by $\delta\Omega$.

A *skeleton* of Ω is a set of line segments (*edges*) S in Ω such that a line segment xy connecting two points x and y exterior to Ω intersects the polygon if and only if xy intersects S.

A *minimum skeleton* for Ω is a skeleton with the least number of edges.

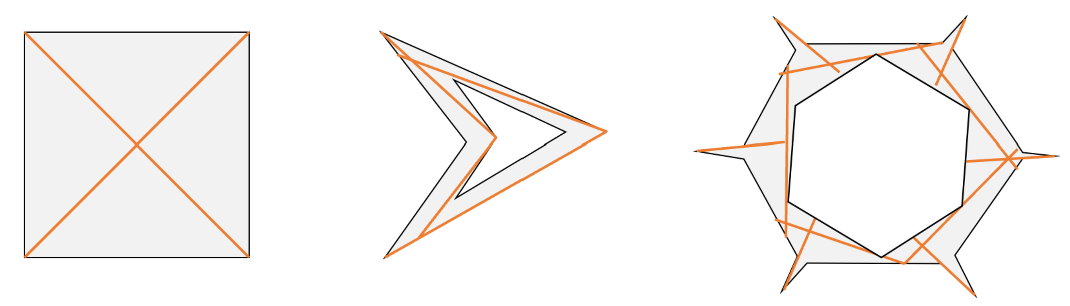

Fig. 1: Examples of minimum skeletons.

Since we are only interested in minimum skeletons, let us assume each skeleton edge is of maximum length, with endpoints on $\delta\Omega$.

Characterisation of Skeletons

We say that a hole h of Ω is *covered* by a set of edges S if any line segment connecting a point x in h and a point y exterior to Ω not in h intersects S.

Proposition: A set of edges S is a skeleton of Ω if and only if it satisfies the following criteria:

1. S meets every convex vertex of Ω.
2. S is connected.
3. Every hole in Ω is covered by S.

Types of Edges in Canonical Skeletons

Proposition: For any polygon, there exists a minimum skeleton in which every edge either meets a concave vertex internally or meets two convex vertices at endpoints.

This allows us to classify all canonical skeleton edges into 2 types :

- A *fixed* edge meets at least two vertices of the polygon.
- A *auxiliary* edge meets exactly one concave vertex internally.

For a polygon Ω with n sides, the number of fixed edges is in $O(n^2)$. To also reduce the number of candidate auxiliary edges to a polynomial in n, we explore their properties further in the next section.

Auxiliary Edges

Let S be a minimum skeleton of polygon Ω with a maximum number of fixed edges. Let e be an auxiliary edge of S, intersecting a vertex z of Ω. We orient the polygon such that e is horizontal with z intersecting from below.

Rotating e counter-clockwise about z until meeting another vertex. Let this resultant fixed edge be $Rot_1(e)$. Similarly define $Rot_2(e)$ for clockwise rotation.

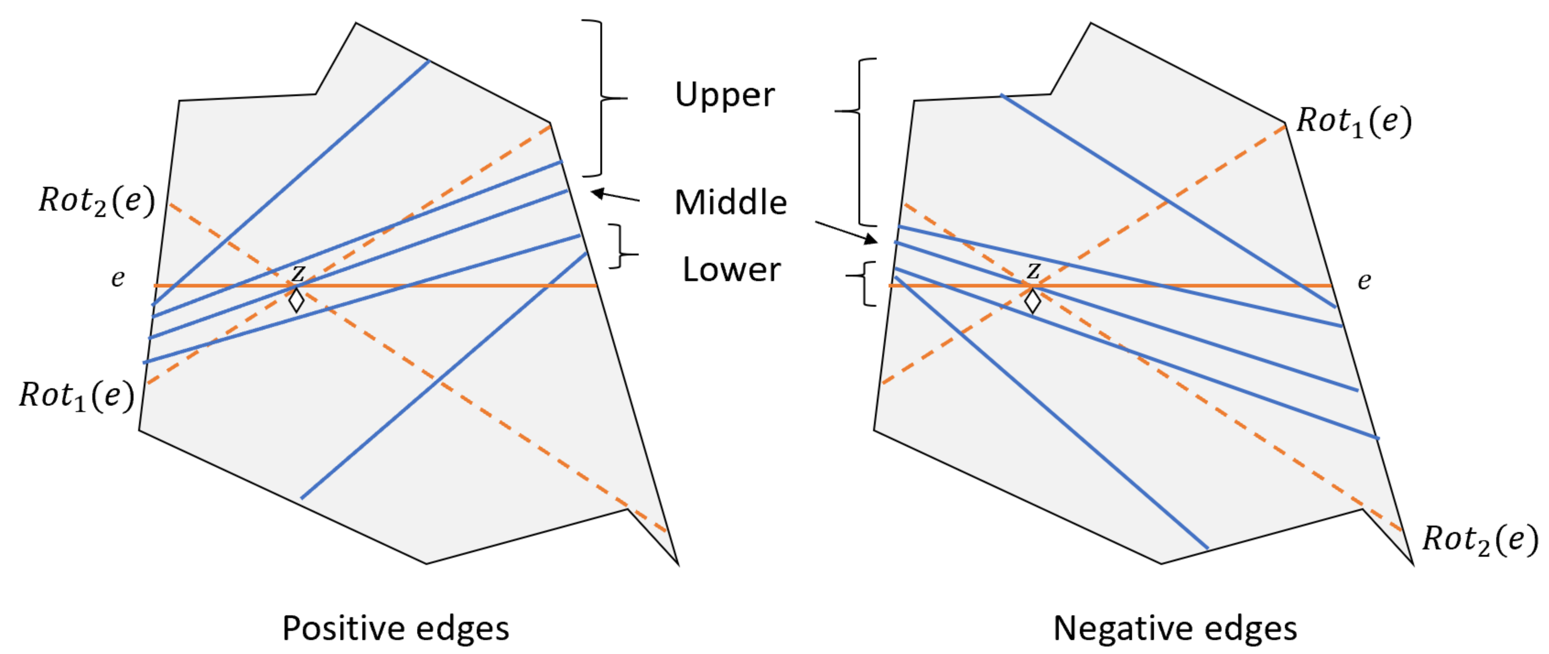

Fig. 2: Positive and negative edges of e

For edges in S that intersect e and the part of $\delta\Omega$ bounded by the rotational extremes:

- Label edges as *positive* or *negative* by their gradient.
- Label them as *upper*, *middle* or *lower* by their position relative to point z.

Proposition: An auxiliary edge either has

- a positive lower edge and a negative lower edge, with no upper edges, or
- a positive upper edge and a negative upper edge, with no lower edges.

Next Step

It has been shown that there exists a canonical minimum skeleton for any simple polygon [1], consisting of fixed edges and auxiliary edges, where auxiliary edges form a path adjacent to a fixed edge (*anchored* path).

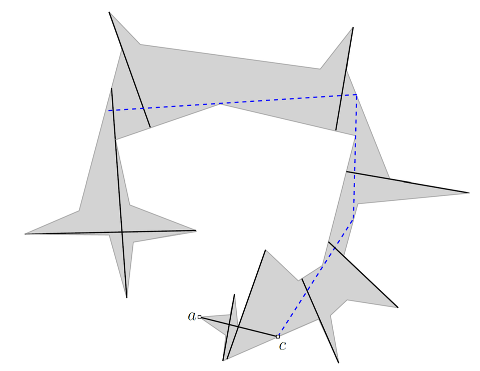

Fig. 3: An example of a canonical minimum skeleton for a simple polygon [1].

To develop a similar canonical minimum skeleton for polygons with holes, further research is needed to investigate:

- Does the process of locking the endpoint of each auxiliary edge to the endpoint of another edge always terminate?
- How to select minimum skeletons from candidate edges using existing integer linear programming algorithms?

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References

[1] N. Andrés-Thió, M. Brazil, C. Ras, D. Thomas and M. Volz. "An Exact Algorithm for Constructing Minimum Euclidean Skeletons of Polygons" [preprint].