Solving the Phase Retrieval Problem

Netanya Robinson with supervisor Matthew Tam

Background

The Phase Retrieval problem involves reconstructing the phase of a signal, which is often lost in physical measurements where only the intensity is recorded.

This problem can be mathematically described with matrices: Find a matrix $\boldsymbol{M}$, when given the complex matrix $|\widehat{\boldsymbol{M}}|$ (the element-wise absolute value of the Fourier transform of $\boldsymbol{M}$) and some additional information about $\boldsymbol{M}$. The additional information might include where $\boldsymbol{M}$ is nonzero (called the support constraint) or that $\boldsymbol{M}$ has positive real entries.

The difficulty of this problem arises from the lack of an inverse function to find $\boldsymbol{M}$ from $|\widehat{\boldsymbol{M}}|$. In general, there are also infinitely many matrices that have the same value as $|\widehat{\boldsymbol{M}}|$.

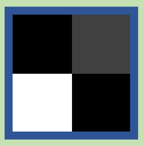
The original matrix $\boldsymbol{M}$ is reconstructed by choosing some initial matrix and iterating projections onto the constraints. Projection meaning the nearest point projection – for example, projection onto a set C is:

$$P_c(x) = \{y \in C : \ |x - y| \leq |x - c| \text{ for all } c \in C\}$$

This produces a sequence which ideally converges to a point that satisfies the constraints simultaneously.

My Project

My project was to compare the various methods used in solving the feasibility version of this problem – to find a solution that satisfies the constraints. We first generate a problem. We start with a grayscale image and represent the image as a matrix of numbers between 0 (black) and 1 (white) for each pixel and then take the element-wise absolute value of the Fourier transform of this image matrix:



$$\rightarrow \begin{bmatrix} 0 & 0.5 \\ 1 & 0 \end{bmatrix} \rightarrow \left| \mathcal{F} \begin{bmatrix} 0 & 0.5 \\ 1 & 0 \end{bmatrix} \right|$$

Image — Image Matrix, $\mathbf{M}$ — $|\widehat{\mathbf{M}}|$

Given $|\widehat{\mathbf{M}}|$ and the aforementioned additional information, we must reconstruct the initial image.

Vacation Scholarship Experience

I really enjoyed and would highly recommend the vacation scholarship. It was a great introduction to mathematics research in an interesting area of my choice. Not only did I learn about phase retrieval, but also many related areas – such as scientific computing and convex optimisation. Thanks to my supervisor Matthew Tam for his extremely helpful guidance and to the vacation scholarship team for this opportunity.

Results

Comparison between different algorithms

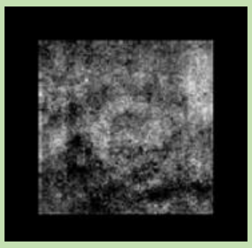
I looked at three algorithms for solving this problem. In general, Fineup's Hybrid Input-Output (HIO) algorithm gave the best reconstructions by far, as demonstrated by the image on the right and its reconstructions below.

Error Reduction Algorithm

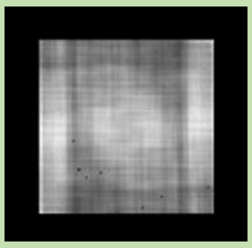
Fineup's Basic Input-Output Algorithm

Fineup's Hybrid Input-Output Algorithm

Even with a very detailed image, like the one to the left, Fineup's HIO algorithm's reconstruction is almost perfect, while the others give a poor reconstruction or fail to converge to a desired point, respectively.



Error Reduction Algorithm

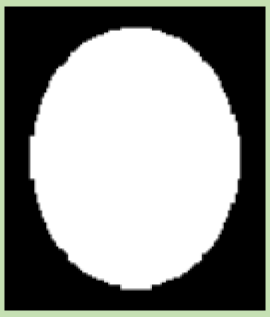


Fineup's Basic Input-Output Algorithm

Fineup's Hybrid Input-Output Algorithm

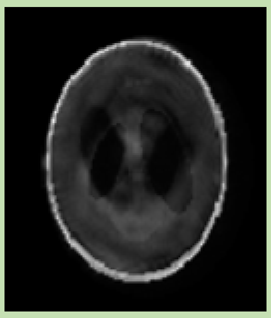
Reconstruction with imperfect supports

The above reconstructions used perfect data on where the image matrix was nonzero. Black borders were even added to provide the support constraint more information. Using an ellipse *approximation* for the support constraint (shown below with the original image and Fineup's HIO algorithm's reconstruction):



Approximation of support (nonzero areas)

Initial image



Fineup's HIO Algorithm

While the reconstruction is poorer, we can also see overlapping copies of rotated or reflected versions of the initial image in the reconstruction - a phenomenon known as the "Twin Problem" in the imaging community.