Stokeslet modelling of the hydrodynamics of stirred nutrients & tracer particles

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Introduction

- How marine microorganisms navigate towards food is a complex topic currently under investigation.
- By stirring dishes of water containing nutrients and microbes, we can gain insight into bacteria's behaviour in aquatic environments [1].
- The flow fields created by this stirring are complicated, and currently found through tracer particle velocimetry (TPV) – a computationally involved method subject to significant noise [1,2].
- This project aimed to develop a simplified model of the flow field created by an arbitrarily moving cylindrical stirring rod, which accurately predicts how nutrients are mixed into the dish over time

The Stokeslet Model

The vector field created by the rod is given by [3]:

$$\boldsymbol{u}^{PF} = \left(\frac{\boldsymbol{I}}{r} + \frac{\boldsymbol{x}\boldsymbol{x}}{r^3}\right) \cdot \frac{\boldsymbol{F}^e}{8\pi\mu}$$

- Where $\boldsymbol{u}^{PF}$ describes the velocity of a particle at $\boldsymbol{x}_p$ in the lab frame.
- $\boldsymbol{x}$ describes the displacement of the particle from the rod at $\boldsymbol{x}_R$ ($\boldsymbol{x} = \boldsymbol{x}_p - \boldsymbol{x}_R$).
- μ is the dynamic viscosity of the mixture.
- r is the distance between the rod and particle ($r = |\boldsymbol{x}|$).
- $\boldsymbol{F}^e$ describes the force of the rod on the water, and is taken to be proportional to the velocity of the rod (since it is the reaction force of the rod's drag), ($\boldsymbol{F}^e = 8\pi\mu A\dot{\boldsymbol{x}}_R$, for some effective drag constant A).

This model creates a vector flow field known as a stokeslet; qualitatively similar in shape to the fields found in experiments using TPV, in the rod's frame of reference (see figure 1).

Assumptions:

- Stokes-Flow (Reynolds Number ≈ 0)
- Diffusion (generated by thermal fluctuations) is negligible
- The cylindrical rod has an infinite height and infinitesimal radius (point force)
- The dish of water is infinite in size (ignore boundary conditions)
- Sedimentation is negligible and particles move only in the xy plane (gravity is ignored)

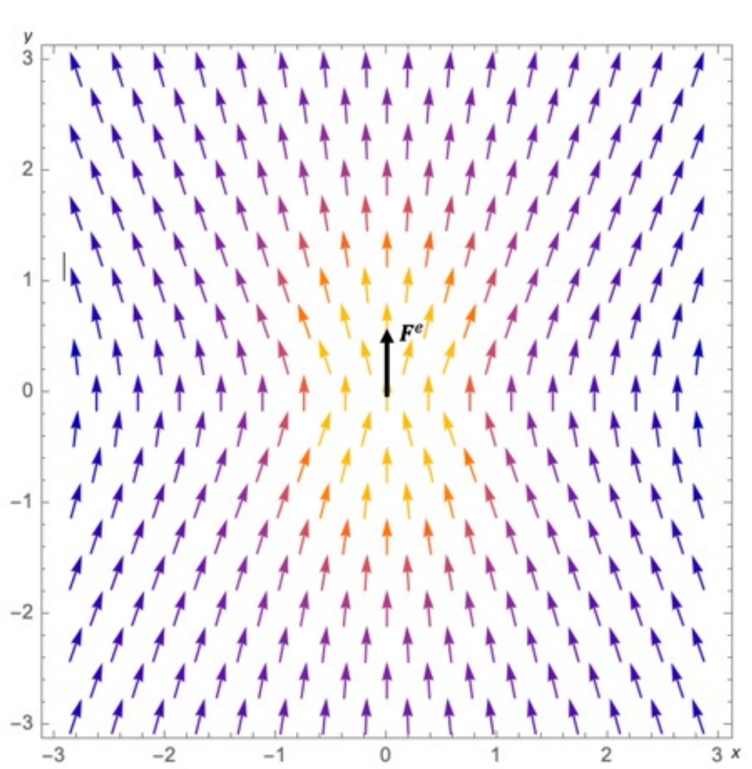

Figure 1. *Vector field of rod (positioned at origin) in the rod frame under the stokeslet model*

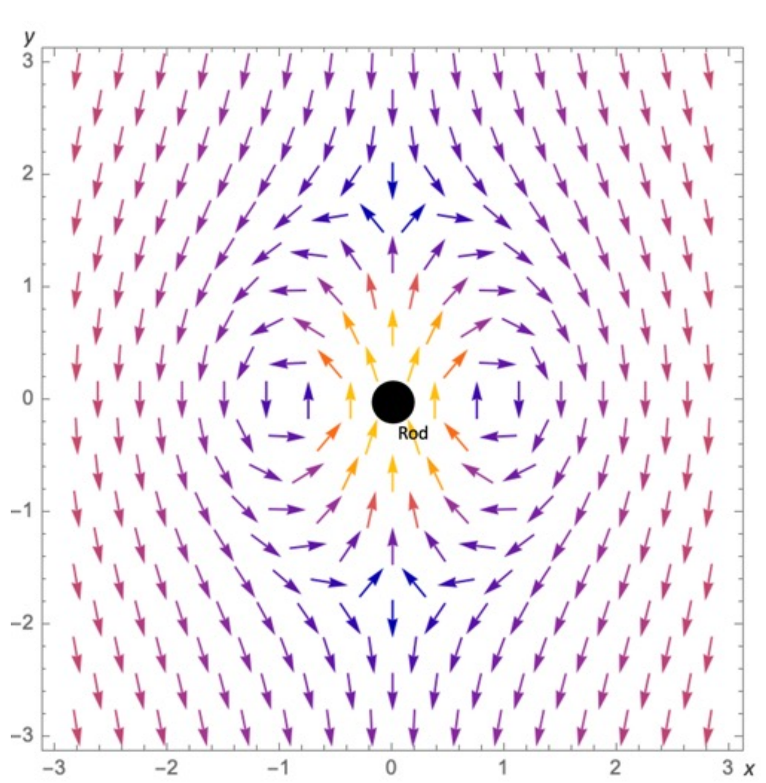

Figure 2. *Vector field of rod under the stokeslet model in the rod's frame*

Straight Line Motion

- Basic case where rod has velocity $\dot{\boldsymbol{x}}_R = \boldsymbol{V} = (0, V)$ for some constant V, and begins at the origin - in the lab frame $\boldsymbol{x}_R(t) = (0, Vt)$.
- Particles begin at $\boldsymbol{x}_{p,0} = (x_0, y_0)$, where $r_0 = |\boldsymbol{x}_{p,0}|$.
- At $\boldsymbol{x}_{p,0} = (\pm A, 0)$, particles run parallel to rod: $\boldsymbol{x}_p(t) = (\pm A, Vt)$.
- Most fruitful analysis occurs in the rod's frame of reference, where the vector field $\boldsymbol{u}_{RF}$ is given by $\boldsymbol{u}_{RF} = \boldsymbol{u}^{PF} - \boldsymbol{V}$, and $\boldsymbol{x}_R(t) = \boldsymbol{0}$ (see figure 2).

In the rod's frame of reference, a given particle will follow the streamline (for $x_0 \neq 0$):

$$y(x) = \pm x\sqrt{\frac{4A^2 r_0^2 x^2}{[r_0 x^2 - x_0^2(r_0 - 2A)]^2} - 1}$$

The kind trajectory of a particle is sensitive to its starting position (see figure 3):

- If it begins with $r_0 \leq 2A$, it will always remain within the circular region $r \leq 2A$ (the 'vortex') (red streamlines).
- However, if $r_0 > 2A$, then it will move around the rod and be left behind (green streamlines).
- The vortex phenomenon is unphysical and due to the point force assumption; however, the circular boundary shows potential utility in modelling the cylindrical rod thusly.

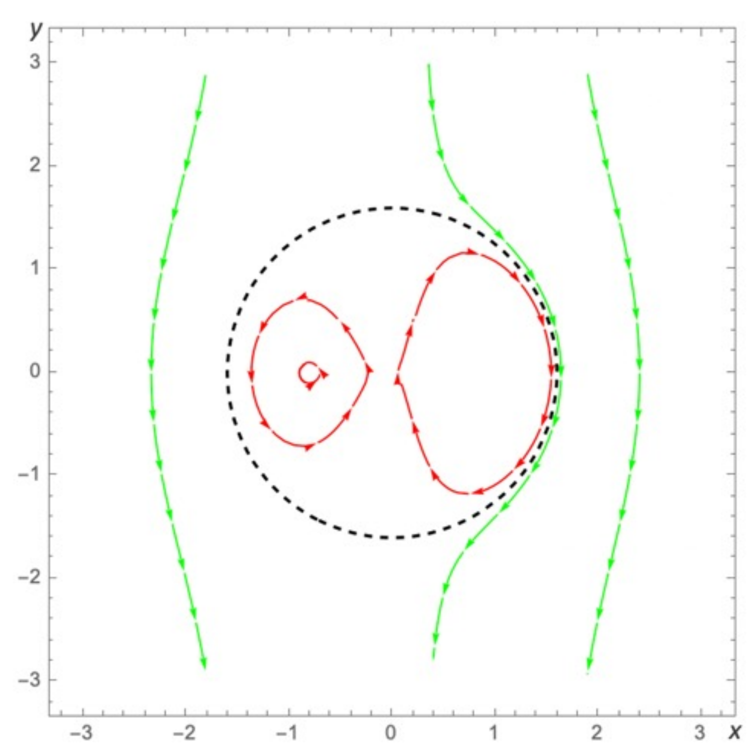

Figure 3. *Streamlines of particles in the rod's frame of reference, both inside (red) and outside (green) the 'vortex' (black).*

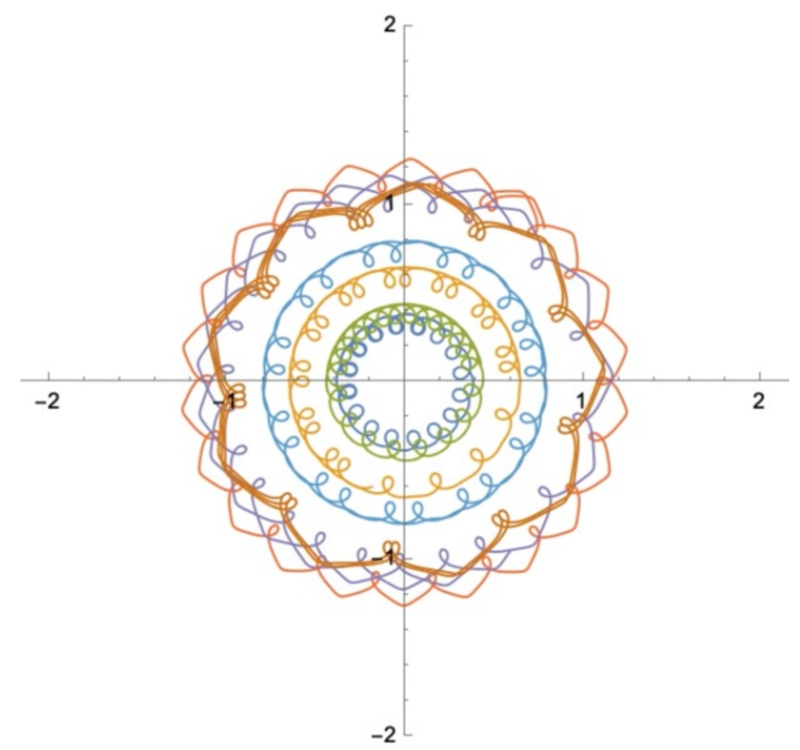

Figure 4. *'Spirographic' trajectories of particles with ongoing circular stirring.*

Circular motion

- Rod travels in a circle of radius R beginning at $\boldsymbol{x}_{R,0} = (R, 0)$ with angular velocity ω; in the lab frame $\boldsymbol{x}_R(t) = (R\cos\omega t, R\sin\omega t)$.
- Particles follow 'spirographic' patterns (see figure 4).
- For a circular path of large enough radius ($R > 4A$), there are 5 initial particle positions ($\boldsymbol{x}_{p,0}$) for which the particle trajectory $\boldsymbol{x}_p$ will also be circular with angular velocity ω ('equilibrium points')
- Namely: $\boldsymbol{x}_{p,0} = \left(R - \frac{8A^2}{\omega^2 R^3}, \pm\frac{2A\sqrt{\omega^2 R^4 - 16A^2}}{\omega^2 R^3}\right), \left(\frac{R}{2} \pm \frac{\sqrt{R^2 - 4AR}}{2}, 0\right)$ and $\left(\frac{R}{2} + \frac{\sqrt{R^2 + 4AR}}{2}, 0\right)$.
- The circularity of all these points' trajectories is unstable, meaning in any physical system with Re > 0 their trajectories would be slightly non-circular.

Figure 8 Motion

- A common stirring pattern used to mimic marine environments is a figure 8 / infinity sign rotation that processes by some angle each circuit.
- From qualitative experimental analysis, particle clumps become 'streaky' when the rod is stirred through them (see figure 5).
- It was found that a drag coefficient of the order $A \approx 0.015$ achieves this phenomenon.
- Diffusion effects were additionally modelled by updating the flow field $\boldsymbol{u} = \boldsymbol{u}^{PF} + B \cdot \boldsymbol{\xi}(t)$, where $\boldsymbol{\xi}(t)$ is a Gaussian white noise function that mimics thermal fluctuations, and the amplitude B is to be experimentally determined (see figure 7).

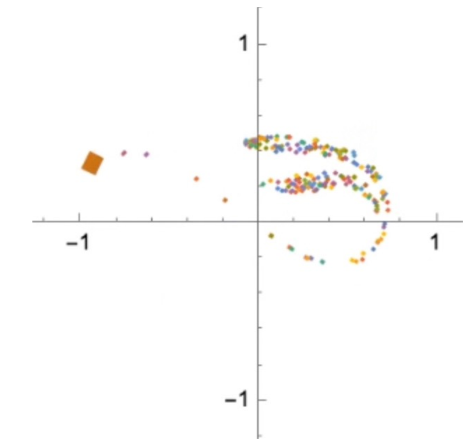

Figure 5. *Streaky particle clump after figure 8 stirring .*

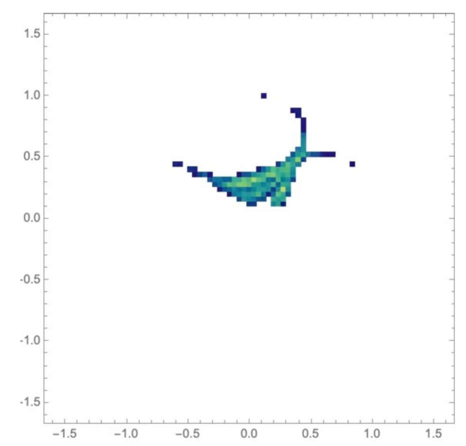

Figure 6. *Binned figure 8 stirring data visualisation.*

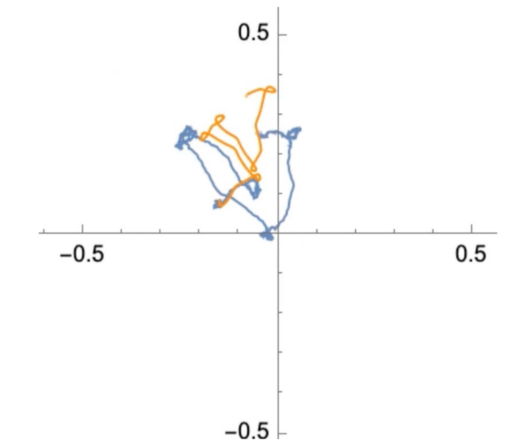

Figure 7. *Single particle trajectory with (blue) and Without (orange) thermal motion.*

Numerical Solutions

- The initial Newtonian Forward-Step method employed was, although computationally fast, found to be inaccurate (relative to analytical solutions in 1 and 2 dimensions) when $\boldsymbol{x}$ is small (the rod is close to the particles).
- Mathematica's 'NDSolve' function was extremely accurate, and the most time efficient solver for large systems of particles
- The Runge-Kutta method (4th order) was also high in accuracy for a reasonable time-step, and was more compatible with the addition of thermal fluctuations, however, was less time efficient for large systems of particles.

Further Research

- One could use laboratory TPV data, and the binned data from the model to find values of best fit for the drag coefficient A (see figure 6).
- This could also be done for the magnitude of thermal fluctuations B, however, given that previous research suggests that diffusive effects do not significantly affect large systems of particles, one would expect B to be small [1].
- This model could also be used to investigate how microbial swimmers (whose swimming mechanisms themselves can be modelled using stokeslets) navigate towards food in marine environments, with additional improvements added.

Straight Line Motion Video.
URL: https://youtu.be/nduvOHwSlY4

Circular Motion Video.
URL: https://youtu.be/bgW_1pzq90M

Figure 8 Motion Video.
URL: https://youtu.be/-2XMc8Bl-5s

Acknowledgement

Firstly, I'd like to thank my supervisor Dr. Douglas Brumley for his time, guidance and support throughout the project. The project gave me fantastic insight into what mathematical research looks like, and how it differs from university study, and has reaffirmed my goal of completing post-graduate studies in mathematics/physics.
I would also like to thank the School of Mathematics and Statistics for providing this program, and especially Dr. Thomas Quella and Dr. Wei Huang for organising many social events and workshops. I would absolutely recommend the program to any student interested in mathematics and research.

References

1. Dunkel, J., Putz, V.B., Zaid, I.M. and Yeomans, J.M. (2010). Swimmer-tracer scattering at low Reynolds number. Soft Matter, 6(17), p.4268.
2. Jeanneret, R., Pushkin, D.O. and Polin, M. (2019). Confinement Enhances the Diversity of Microbial Flow Fields. Physical Review Letters, 123(24)