Initial steps towards determining lava flux from front position

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Introduction

Modelling lava flows is of interest to those living in the vicinity of volcanos, given their destructive potential. The flow of lava down a volcano can be modelled at its simplest as the gravity current of a viscous fluid on an inclined plane[1]. For simple fluxes we end up with similarity solutions for the flow. For more complicated fluxes the method of characteristics can be employed.

Our governing equation

$$\frac{\partial h}{\partial t} + \frac{\partial h^3}{\partial x} = 0 \tag{1}$$

is drawn from Lister[2], and is an example of a generalised inviscid Burgers' equation.

Similarity Solutions

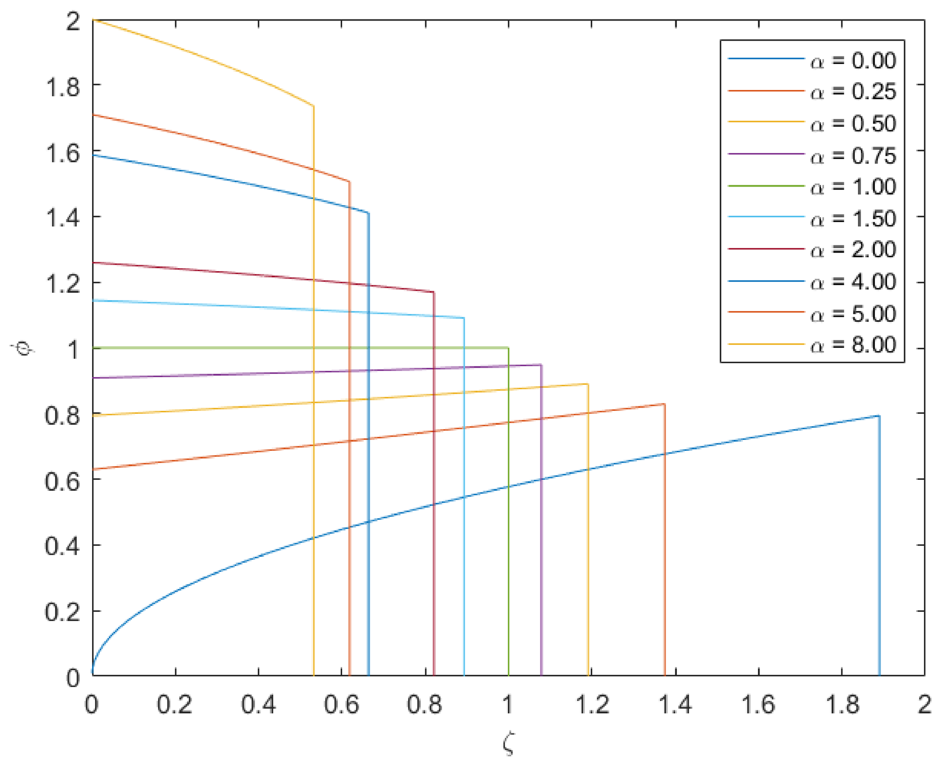


Fig. 1: Similarity solutions, adapted from Lister[2] Fig. 2

In simple situations we may reduce our partial differential equation into an ordinary differential equation in some new dimensionless variable, η.
Suppose we introduce a few power laws, for h and x and volume.

$$h \sim t^a \tag{2}$$
$$x \sim t^b \tag{3}$$
$$V \sim t^{a+b} \tag{4}$$
$$\sim t^{\alpha} \tag{5}$$

Combining with our governing equation and solving the resulting system of linear equations we get for the powers, we can write

$$h = ct^a = t^a f(\eta) = t^{\frac{\alpha-1}{3}} f(\eta), \ \eta = \frac{x}{t^b} \tag{6}$$

Thus producing

$$af(\eta) + (3f^2(\eta) - b\eta)\frac{\partial f}{\partial \eta} = 0 \tag{7}$$

Equation 7 can be solved with an integrating factor of $f^{-\frac{3a}{a-1}}$. Plotting the solutions, keeping in mind conservation of volume, produces Figure 1.

Fig. 2: Ekrem Canli, CC BY-SA 4.0 <https://creativecommons.org/licenses/by-sa/4.0>, via Wikimedia Commons, https://commons.wikimedia.org/wiki/File:K%C4%ABlauea_lava_flow_2017(2).jpg

Fig. 3: Fissure 8 on Kīlauea, courtesy of the U.S. Geological Survey, https://www.usgs.gov/media/images/fissure-8-kilauea

The Inverse Problem

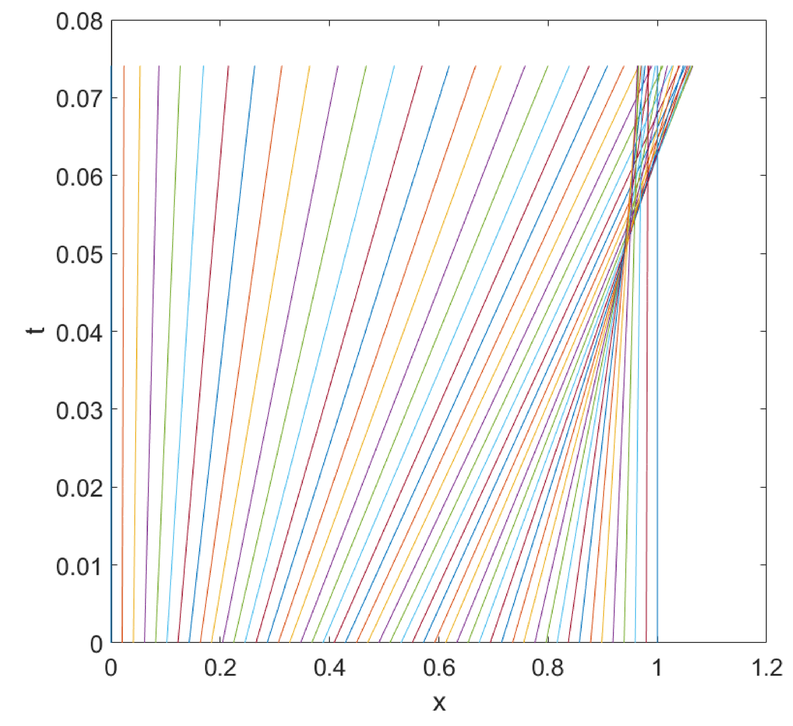
The position of the front as a function of time can be found for any given flux as a function of time, by applying our methods of characteristics approach. The question then is whether the reverse may be true. Simply running time backwards will blow our equation up, a la the inverse heat equation, but a physical-esque method of reversing the flow seems entirely plausible, but has not as of yet been developed.

References

[1] J.E. Simpson, Gravity Currents: In the Environment and the Laboratory. (E. Horwood, 1987).
[2] J.R. Lister, Journal of Fluid Mechanics 242, 631 (1992).
[3] S.J. Farlow, Partial Differential Equations for Scientists and Engineers (Wiley, New York, 1982).

Many thanks to Dr. Edward Hinton for much valuable guidance and discussion

Method of characteristics



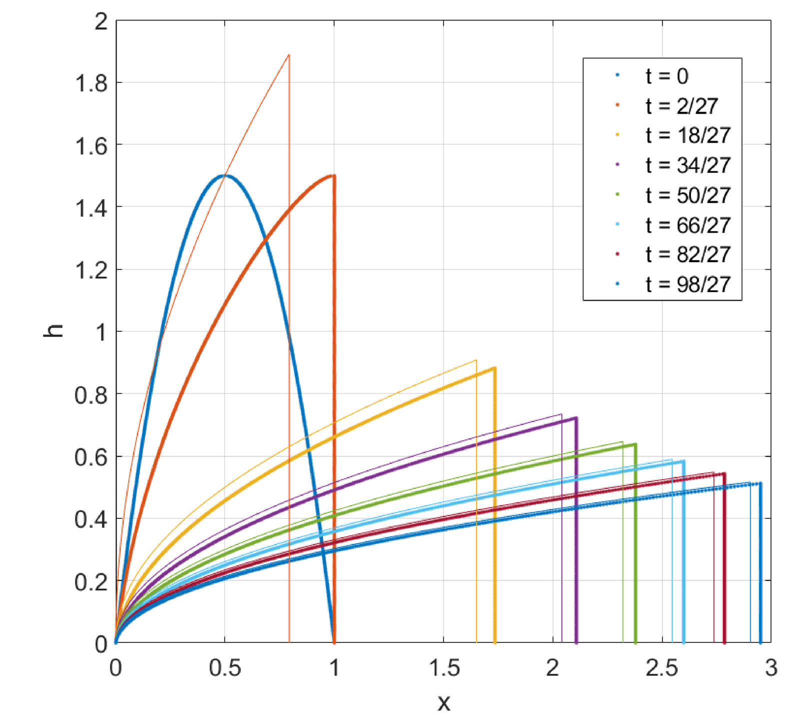


Fig. 4: Characteristics for parabolic initial conditions, note the crossed characteristics indicating a shock

Fig. 5: Height at various times, for parabolic initial conditions

For more complicated flows, which we'll need to be able to tackle given the desire for a general tool, we'll need a different approach. I've used the method of characteristics. Farlow[3] offers a good introduction to the technique.
We consider the perspective of a moving observer, so we have x(t), such that the flow appears fixed in place.

$$\frac{\partial h}{\partial t} + 3h^2\frac{\partial h}{\partial x} = 0 \rightarrow \frac{dh}{ds} = \frac{\partial h}{\partial t}\frac{dt}{ds} + \frac{\partial h}{\partial x}\frac{dx}{ds} = 0 \tag{8}$$

So $\frac{dt}{ds} = 1$ and $\frac{dx}{ds} = 3h^2$. Integrating, we can choose to add an extra parameter to t or x, for boundary or initial conditions respectively. Initial conditions are somewhat easier, so we write $t = s$ and $x = 3h^2 s + \tau$. For initial conditions we consider

$$h(x,0) = \begin{cases} 0 & x < 0 \\ 6(x - x^2) & 0 \le x \le 1 \\ 0 & x > 1 \end{cases} \tag{9}$$

This has the nice property of having a unit volume. Figure 4 is simply produced by plotting x against t for a range of τ values.

Figure 5 is the product of a somewhat more involved process. Knowing that h is constant along characteristics, it's simply a function of τ. But, τ can be written as a function of x and t. Thus we can see how height varies over time, with the caveat that height will often end up multi-valued, which is clearly non-physical. To address this we insert shocks, discontinuities in height, that move over time such that volume is conserved. In our constant volume case that means ensuring the volume remains one. Shock position may then be determined by the equation

$$1 = h_s x(h_s) - \int_0^{h_s} x(h)dh \tag{10}$$

We can solve this numerically without difficulty, and that gives us Figure 5. Note this is the long-time solution to the flow, after the shock has reached the front. Happily, this matches our constant volume similarity solution very well, unhappily, we've not determined shock position before it reaches the front, for which we'll need to delve into Rankine–Hugoniot conditions.