Super curves

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Introduction

Algebraic geometry describes the relationship between algebra and geometry.
In this project, we explore how geometric problems translate to algebraic problems, and we introduce the notion of Grassmann algebras to explore how the answers to geometric problems differ in anti-commutative spaces.

Grassmann algebras

Let ζ^a, $a = 1, \ldots, N$, be a set of generators for an algebra, which anticommute:

$$\zeta^a \zeta^b = -\zeta^b \zeta^a, \quad (\zeta^a)^2 = 0 \quad \text{for all } a, b.$$

The algebra is called a *Grassmann algebra* [4] and is denoted by Λ_N. We deal with the formal limit $N \to \infty$; the corresponding algebra is denoted by Λ_∞. Note that the elements $1, \zeta^{a_1}, \zeta^{a_1}\zeta^{a_2}, \ldots$, where the exponents within each product range over all finite sequences of strictly increasing integers, form an infinite basis for Λ_∞. The elements of Λ_∞ will be called *supernumbers*. Every supernumber can be expressed in the form $z_B + z_S$, where the *body* z_B is an ordinary complex number, and the *soul*

$$z_S := \sum_{n=1}^{\infty} \sum_{a_1, \ldots, a_n} c_{a_1 \ldots a_n} \zeta^{a_1} \cdots \zeta^{a_n}.$$

Restricting over odd indices n gives an *odd* supernumber, and vice versa for an *even* supernumber.

Bezóut's theorem

Here we state Bezóut's theorem for the case of plane curves, see [1].
Suppose that X and Y are two plane projective curves defined over an algebraically closed field k that do not share a nonconstant factor. Then the total number of intersection points of X and Y in $\mathbb{P}^2_k$ counted with their multiplicities, is equal to the product of the degrees of X and Y.

Complex supermanifolds

DeWitt [4] defines a complex *supermanifold of dimension* (m, n) to be a space M together with a collection of ordered pairs $(\mathscr{U}_A, \phi_A)$ where each $\mathscr{U}_A$ is a subset of M, and its associated ϕ_A is a one-to-one mapping of $\mathscr{U}_A$ onto an open set in $\mathbb{C}^{m|n} := \mathbb{C}_c^m \times \mathbb{C}_a^n$ (in the coarse sense). Here, $\mathbb{C}_c$ are the even supernumbers and $\mathbb{C}_a$ are the odd supernumbers. The collection of ordered pairs is required to have the following properties:

(1) $\bigcup_A \mathscr{U}_A = M$

(2) $\phi_A \circ \phi_B^{-1}$ is differentiable for all nonempty intersections $\mathscr{U}_A \cap \mathscr{U}_B$.

Keßler [3] defines a complex supermanifold $\mathbb{C}^{m|n}$ to be the topological space $\mathbb{C}^m$ together with the sheaf $\mathcal{O}_{\mathbb{C}^{m|n}} = \mathcal{H}_{\mathbb{C}^m} \otimes_{\mathbb{C}} \bigwedge_n^{\mathbb{C}}$. The equivalence of the definitions of DeWitt and Keßler is non-trivial; it uses the functor of points.
Using this definition, algebraic curves can now be embedded as complex supermanifolds.

Degree of algebraic maps between curves

Let C be a curve $(p(x, y) = 0) \subset \mathbb{A}^2_k$, where p is a polynomial of genus greater than 0 over an algebraically closed field k. Let $f : C \to \mathbb{A}^1$ be any nonconstant polynomial map.

Via composition, f induces a ring homomorphism $\phi : k[t] \to k[x, y]/p(x, y)$. Now using Hilbert's Nullstellensatz [2], the maximal ideals of $k[t]$ are of the form $(t - a)$, and represent points in $\mathbb{A}^1$. Using Bezóut's theorem on the corresponding projective curves $(P = 0), (F_a = 0) \subset \mathbb{P}^2_k$, we then discover that for all but finitely many maximal ideals $m \subset k[t]$, the ideal generated by $\phi(m)$, i.e. $\langle \phi(m) \rangle$ is contained in a constant number d of maximal ideals known as the *degree* of the map.

Specifically, Bezóut's theorem implies the number of intersections of P and F_a counting multiplicities is $D_P \cdot D_{F_a}$, where D_P and D_{F_a} are the degrees of P and F_a respectively. Then $d = D_P \cdot D_{F_a} - T$ for $a \in \mathbb{A}^1 \setminus (A \cup B)$, where A is a finite set of points on which F_a and P share a nonconstant factor (and thus may have infinitely many common roots), and B is a finite set on which there exists a common root of P and F_a with multiplicity greater than 1. Here, T is a constant indicating the common roots "at infinity" (i.e., the common roots added by converting to the projective curves P and F).

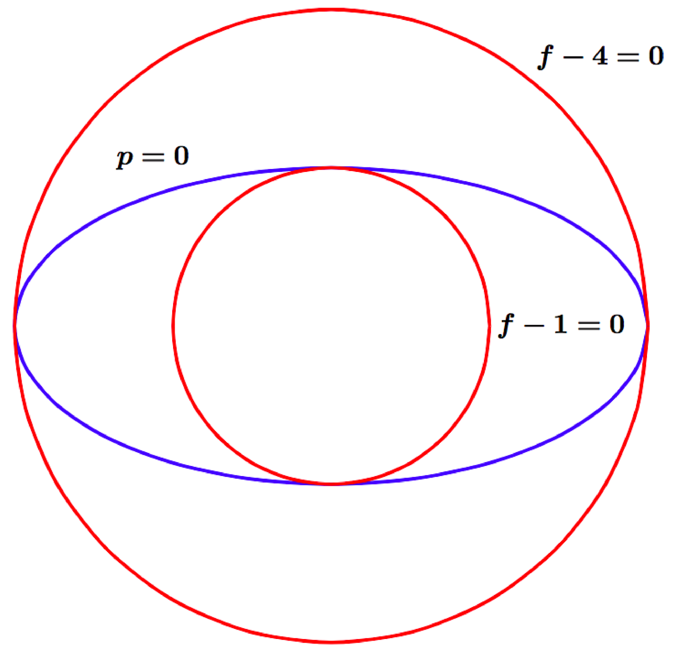

Figure 1: $a = 1, 4$

The graph above depicts the (real) intersections of the ellipses $p(x, y) = x^2/4 + y^2 - 1 = 0$ and $f(x, y) - a = x^2 + y^2 - a = 0$ over $k = \mathbb{C}$ for $a \in \{1, 4\} \subset B$, i.e. the values of a resulting in tangency. Here $A = \varnothing$ and $T = 0$ (there are no common roots at infinity).

Degree one maps

Suppose $d = 1$ in the context of the above discussion. Then f induces a holomorphic map $C \to \mathbb{P}^1_k$ that is locally invertible with holomorphic inverse. Since this map is trivially surjective, we have the homeomorphism $C \cong \mathbb{CP}^1$, meaning the genus of C must be 0.

This has some geometric intuition; the geometric genus for plane algebraic curves is analogous to topological genus, so if C has genus greater than 0, it contains a "closed loop". But this would have more than $d = 1$ intersections with $f(x, y) - a$ for infinitely many values $a \in \mathbb{A}^1$, a contradiction.

The super case and conclusions

Above we have converted geometric statements to algebraic statements, such as the definition of the degree of a map. Why is this important?
It enables us to study the analogues of such statements in the super case which are necessarily in algebraic form, since geometric intuition is often useless in anti-commutative spaces.
For example, suppose $p \in \mathbb{C}[x, y]$ but we consider the locus $C : (P = 0) \subset \mathbb{CP}^{2|1}$ of the corresponding projective polynomial. One could pose the question: if every point in the codomain of a nonconstant polynomial map $F : C \to \mathbb{CP}^{1|1}$ has exactly one preimage, is it true that $C \cong \mathbb{CP}^{1|1}$?
Understanding the equivalent algebraic formulation is nontrivial as the Nullstellensatz does not apply for noncommutative rings, but alternative approaches that worked in the non-super case (e.g. using Bezóut's theorem or the holomorphicity of F) may be infeasible.

References

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