THE INVERSE GALOIS PROBLEM OVER $\mathbb{C}(t)$

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INTRODUCTION

The study of Galois Theory is concerned with the symmetries that arise in finite extensions of fields. The set of all automorphisms that fix the base field form a group and under certain conditions, if this group is 'large' enough then it is called a Galois group and the extension is called Galois. In particular one looks at and can often compute the Galois Groups given an arbitrary Galois extension. However a natural question to ask is whether given a field $\mathbb{F}$ and an arbitrary group G, does there exist a finite Galois extension with Galois group equal to G? Such a question is called the **Inverse Galois Problem**. In complete generality this is not the case, for example all non-cyclic groups do not appear as the Galois Group of any extension of the finite fields. However the largest open problem is whether this is true over $\mathbb{Q}$. We instead opt to investigate the solved problem; whether all finite groups appear as a Galois group of an extension of $\mathbb{C}(t)$ which enables us to use the powerful tools of Complex Analysis to construct such an extension.

RIEMANN SURFACES

A **Riemann Surface** X is a Hausdorff topological space which is locally isomorphic to the complex plane. Slightly more concretely this means that any point in X has a neighbourhood that can be mapped biholomorphically onto an open set in $\mathbb{C}$ such as the unit disk.

Formally a **finite cover** of a Riemann Surface is a function from one Riemann Surface to another, $p : Y \to X$ such that for each point $x \in X$ there exists a neighbourhood U of x such that $p^{-1}(U) = \coprod_{i=1}^{n} V_i$ where each V_i is an open set in Y and the restriction $p|_{V_i} : V_i \to U$ is a homeomorphism. Intuitively Y 'covers' X with n layers which are all in some sense 'copies' of X. Figure 1 helps visualise this.

If X is a Riemann surface then a function $f : X \to \mathbb{C}$ is said to be **meromorphic** if for all charts $z : V_i \to \mathbb{C}$; $f \circ z^{-1} : \mathbb{C} \to \mathbb{C}$ is meromorphic in the usual sense.

We denote the field of all meromorphic functions on X by $\mathcal{M}(X)$. Importantly for our purposes we have that $\mathcal{M}(\mathbb{P}^1(\mathbb{C})) \cong \mathbb{C}(t)$

The **universal cover** $\tilde{X}$ is in some sense the largest cover of X. It has the universal property that for any other connected cover $\pi : Y \to X$ and points $\tilde{x} \in \tilde{X}$ and $y \in Y$ with $p(\tilde{x}) = \pi(y)$ there exists a unique homeomorphism $\sigma : \tilde{X} \to Y$ such that $\sigma(\tilde{x}) = y$.

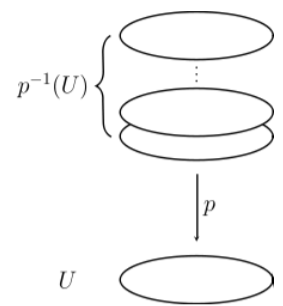**Figure 1:** Visualisation of a covering map.
Image Source: Wikipedia

THE RIEMANN EXISTENCE THEOREM

A natural question to ask is that given a covering Y of a Riemann Surface X why would there exist more meromorphic functions on Y than there does on X? It turns out the answer to this question and many others in this project is the Riemann Existence Theorem. The form of the theorem we rely on is as follows:
For any compact Riemann Surface X with points $x_1, ..., x_n \in X$ and values $a_1, ..., a_n \in \mathbb{C}$ there exists a meromorphic function $f \in \mathcal{M}(X)$ such that $f(x_1) = a_1, ..., f(x_n) = a_n$.
It is this that guarantees the existence of enough meromorphic functions in a cover of X to provide a sizeable extension of the field of meromorphic functions on X.

FUNDAMENTAL GROUPS

In topology the study of how loops (closed paths) behave is an important tool in the understanding of a topological space. Consider two paths f_1 and f_2. If f_1 can be continuously deformed to f_2 then the two loops are called **homotopic** and for our purposes we consider them to be essentially the same path. For example in the diagram on the right one can imagine carefully 'bending' and shifting f_1 until it becomes f_2. The set of all loops rooted at some point $x \in X$ where we consider two loops to be identical if they are homotopic is a group called the **fundamental group** which is denoted by $\pi_1(X, x)$. This group helps encapsulate various properties of the space such as the number of 'holes'.
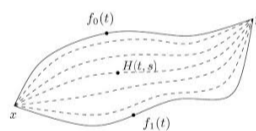

Figure 2: Visualisation of continuously deforming one path to another.
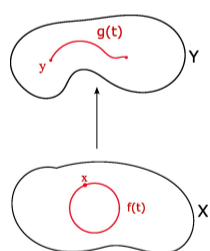**Image Source:** University of Chicago

Figure 3: Visualisation of the lifting of a loop into the covering space

A key tool in our study was the use of a process called a **lift**, which, in the context of covering spaces, takes a loop rooted at some x in the base space and 'lifts' it up to produce a path, starting at some point in the fibre of x, in the covering space. This lift respects the covering map in that the image of the lifting of a path is the original path so in some sense this is an inverse image of the original loop. However one may note that it is not actually the inverse image as the covering map is most likely not invertible so such a preimage is not unique but for our purposes once you specify a starting point for the loop in the fibre the lifting is in fact unique. This allows for a concrete method to compare the fundamental group of the base space to the covering space.

AUTOMORPHISMS OF THE COVERING SPACE

Another key step in our construction is the notion of a fibre preserving automorphism. Intuitively, if $p : Y \to X$ is a covering map then it is an automorphism that 'respects' the covering map, meaning it sends elements of $p^{-1}(x)$ to other elements of $p^{-1}(x)$. With slight abuse of notation we denote the set of all such automorphisms as $\mathrm{Aut}(Y|X)$. More formally an automorphism σ is in $\mathrm{Aut}(Y|X)$ if $p \circ \sigma = p$.
In addition we also wish to study the extension $\mathcal{M}(Y)|\mathcal{M}(X)$. Here this is again a slight abuse of notation as in general $\mathcal{M}(X) \not\subset \mathcal{M}(Y)$ but we instead consider this as an extension of the embedding of $\mathcal{M}(X)$ in $\mathcal{M}(Y)$ with the embedding being given by the function $p^* : \mathcal{M}(X) \to \mathcal{M}(Y)$ defined by $p^*(f) = f \circ p$. Then with this we can define the $\mathrm{Aut}(\mathcal{M}(Y)|\mathcal{M}(X))$ in the more usual way as the set of all automorphisms of $\mathcal{M}(Y)$ which fix the embedding of $\mathcal{M}(X)$ in $\mathcal{M}(Y)$.

A covering map $p : Y \to X$ is considered to be **Galois** if $\mathrm{Aut}(Y|X)$ acts transitively on each fibre $p^{-1}(x)$ for $x \in X$. It is worth noting that the universal covering does satisfy this condition.

FINAL CONSTRUCTION

This summary excludes a massive amount of detail but the general construction of the field extension of $\mathbb{C}(t)$ is as follows:

1. Suppose our desired Galois Group is G. Then using the general fact that there exists $n \in \mathbb{N}$ and $N \trianglelefteq \mathbb{F}_n$ such that $G \cong \mathbb{F}_n/N$, i.e G is the quotient of the free group by some normal subgroup N.

2. Then consider the (n+1)-punctured Riemann Sphere $X = \mathbb{P}^1(\mathbb{C}) - \{x_1, ..., x_{n+1}\}$. This has a fundamental group isomorphic to the free group on n generators $\mathbb{F}_n$.

3. Then take the universal cover which we denote by $\tilde{X}$. This exists since X is connected.

4. Then using the realisation of N as a subgroup of $\mathrm{Aut}(\tilde{X}|X) \cong \pi_1(X, x)$ we consider the quotient space $\tilde{X}/N$ which is just $\tilde{X}$ where two elements $x_1, x_2 \in \tilde{X}/N$ are considered equal if there exists $\sigma \in N$ which sends x_1 to x_2.

5. Then we have that: $\mathrm{Aut}(\tilde{X}/N|X) \cong \mathrm{Aut}(\mathcal{M}(\tilde{X}/N)|\mathcal{M}(X)) \cong \mathbb{F}_n/N \cong G$.

6. Then one can extend the covering map from $\tilde{X}/N$ to X, to a covering map from some Riemann Surface Y to $\mathbb{P}^1(\mathbb{C})$ with the nice property that $\mathrm{Aut}(\mathcal{M}(\tilde{X}/N)|\mathcal{M}(X) \cong \mathrm{Aut}(\mathcal{M}(Y)|\mathcal{M}(\mathbb{P}^1(\mathbb{C})))$

7. Then using the fact that $\mathcal{M}(\mathbb{P}^1(\mathbb{C})) \cong \mathbb{C}(t)$ we see that in fact $\mathcal{M}(Y)|\mathbb{C}(t)$ is the desired field extension with Galois group G.

CONCLUSION

In conclusion, if one is given any arbitrary finite group G then using this construction we can construct a field extension of $\mathbb{C}(t)$ with Galois group G. The only downside to this method is that the desired extension is given in very abstract terms, it is not easy to see what functions such a field extension has that are not in $\mathbb{C}(t)$ even though we know they must exist. In summary though their general existence is assured, giving an explicit solution remains very challenging. Indeed to even compute the universal cover of the twice punctured plane is quite non-trivial, then to give an explicit form for the quotient space required is in general a very challenging problem. However if the reader is content with knowing that such an extension does exist for all finite groups then the general existence argument given is indeed sufficient. Another area to look further into would be understanding the Riemann Existence Theorem. Though a deceptively simple theorem, modern proofs for it require the finiteness of certain cohomology groups which offer a very interesting avenue of further study.

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