Coxeter groups: Examining the relationship between the Imaginary cone and the Tits cone

Tamalyn Davies

Supervised by Lawrence Reeves

THE UNIVERSITY OF MELBOURNE

Introduction

Informally, a **Coxeter group** can be though of as a group that is generated by **reflections**. The study of such groups lends itself nicely to the use of algebraic, geometric and combinatorial perspectives; it finds many applications in various areas of mathematics.

In this poster we explore the general theory of Coxeter groups, placing an emphasis on their associated 'root systems' and geometric representation as a group generated by reflections in a euclidean space (following Humphreys [2]). In doing so, we will work to establish the relationship between the 'Tits cone' and the 'Imaginary cone' under mild finiteness and non-degeneracy conditions.

Coxeter group definition

A **Coxeter group** is a group W that admits a presentation relative to a generating set S, subject only to relations of the following form:

$$(s,s')^{m(s,s')} = 1$$

Where $m(s,s') = 1$ and $m(s,s') = m(s',s) \in \mathbb{N}_{\geq 2}$ for $s \neq s'$ in S. The pair (W,S) is called a **Coxeter system**.

In the case that no relation occurs for the pair s,s', we say by convention that $m(s,s') = \infty$. Although much of what will be discussed holds for arbitrary S, we will always assume S to be **finite**. We note that it can be shown that each $s \in S$ has order 2 in W.

Some examples of Coxeter groups are as follows:

Dihedral groups

For any $n \in \mathbb{N}_{\geq 2}$ The **dihedral group** D_n of order $2n$, is a Coxeter group with the following presentation:

$$D_n = \langle s_1, s_2 \mid (s_1)^2 = (s_2)^2 = 1, (s_1 s_2)^n = 1 \rangle$$

For D_n we see that $m(s_1,s_2) = n$, however, in the case that $m(s_1,s_2) = \infty$, we get the **infinite dihedral group** D_∞.

Triangle groups

For integers l,m,n greater than or equal to 2, the **triangle group** $\Delta(l,m,n)$ is a Coxeter group with the following presentation:

$$\Delta(l,m,n) = \langle s_1,s_2,s_3 \mid (s_1)^2 = (s_2)^2 = (s_3)^2 = 1, (s_1 s_2)^l = (s_2 s_3)^m = (s_3 s_1)^n = 1 \rangle$$

When interpreted geometrically, $\Delta(l,m,n)$ can be though of as the group generated by the reflections in the sides of a triangle with internal angles $(\frac{\pi}{l}, \frac{\pi}{m}, \frac{\pi}{n})$. In turn, each triangle group corresponds to a triangular tiling of either euclidean, spherical, or hyperbolic space.

If $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} = 1$, the corresponding tiling is of the euclidean plane (Figure 1).

If $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} > 1$, we obtain a tiling of the sphere (Figure 2). In this case $\Delta(l,m,n)$ is **finite**.

If $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$, we obtain a tiling of hyperbolic space (Figure 3).

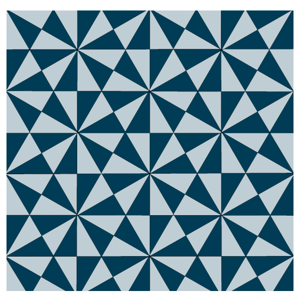

Figure 1. $\Delta(2,3,6)$

Figure 2. $\Delta(2,3,3)$

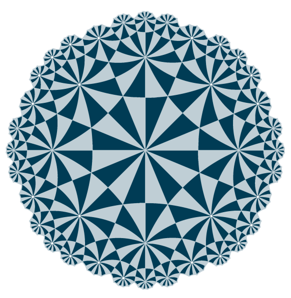

Figure 3. $\Delta(2,3,8)$

The length function

Since the generators $s \in S$ have order 2 in W, each $w \neq 1$ in W can be written in the form

$$w = s_1 s_2 \ldots s_r$$

for some (not necessarily distinct) s_i in S. We define the length $\ell(w)$ of w to be the smallest r for which such an expression exists, and we call this expression ***reduced***. By convention $\ell(1) = 0$.

In the case that all $m(s_\alpha, s_\beta) = \infty$ for $s_\alpha \neq s_\beta$, we call W a **universal** Coxeter group. A Coxeter group W is universal if and only if each $w \in W$ has a unique reduced expression.

We note the following important result: For all $s \in S$ and $w \in W$, $\ell(ws) = \ell(w) \pm 1$. Similarly for $\ell(sw)$.

Geometric representation of W

Given a Coxeter system (W,S), we aim to construct a representation of W as a group generated by reflections in a euclidean space. It is too much to expect these reflections to be orthogonal, but we can construct a suitable alternative by defining a reflection to be as follows:

A ***reflection*** is a linear transformation that sends some nonzero vector to its negative, and fixes pointwise a hyperplane in some vector space V.

Now let V be a vector space over $\mathbb{R}$, having a basis $\{\gamma_s \mid s \in S\}$ in one-to-one correspondence with S. We define a symmetric bilinear form B on V as follows:

$$B(\gamma_s, \gamma_{s'}) = -\cos \frac{\pi}{m(s,s')}$$

where we define $B(\gamma_s, \gamma_{s'}) = -1$ if $m(s,s') = \infty$. For each $s \in S$ we define a reflection $\sigma_s : V \to V$ by the rule:

$$\sigma_s(\lambda) = v - 2B(\gamma_s, \lambda)\gamma_s$$

which fixes pointwise the hyperplane $H_{\gamma_s} = \{\lambda \in V \mid B(\gamma_s, \lambda) = 0\}$.

By this construction we get a unique homomorphism $\sigma : W \to GL(V)$, such that:

$$s \mapsto \sigma_s$$

and the group $\sigma(W)$ preserves the form B on V. Importantly, we get that the representation $\sigma : W \to GL(V)$ is **faithful** [2, page 113].

Root systems

For ease of notation, we will write $w(\gamma_s)$ in place of $\sigma(w)(\gamma_s)$.

The **root system** Φ of W is defined to be the collection of all vectors $w(\gamma_s)$ in V, where $w \in W$ and $s \in S$. These vectors are unit vectors, since the group $\sigma(W)$ preserves the form B on V.

If α is any root, we can write it uniquely in the form:

$$\alpha = \sum_{s \in S} c_s \gamma_s \ (c_s \in \mathbb{R})$$

where the c_s coefficients are all of like sign. We call a root α **positive** and write $\alpha > 0$ if all $c_s \geq 0$. Similarly, call a root α **negative** and write $\alpha < 0$ if all $c_s \leq 0$. Let Φ^+ and Φ^- denote the respective sets of positive and negative roots.

The infinite dihedral group D_∞ has an infinite root system, a portion of which is depicted below:

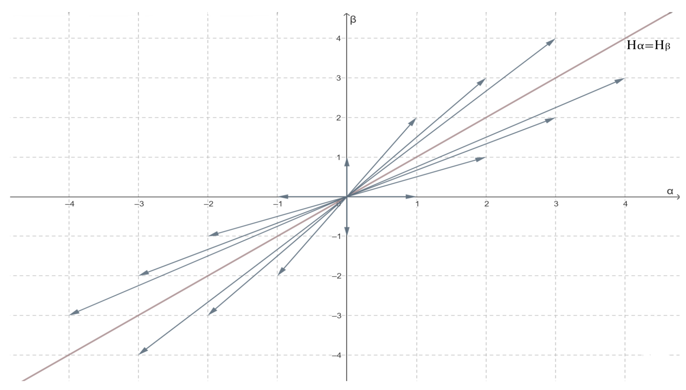

Figure 4. Root system for $D_\infty = \langle s_1, s_2 \mid (s_1)^2 = (s_2)^2 = 1 \rangle$

Theorem [2]: *Let $w \in W$ and $s \in S$. If $\ell(ws) > \ell(w)$, then $w(\gamma_s) > 0$. If $\ell(ws) < \ell(w)$, then $w(\gamma_s) < 0$.* □

We also note that for any $w \in W$, $\ell(w)$ is equal to the number of positive roots sent by $\sigma(w)$ to negative roots [2].

Roots and reflections

For each root $\alpha \in \Phi$, there is an associated reflection in $GL(V)$. Suppose $\alpha = w(\gamma_s)$ for some $w \in W$ and $s \in S$. A brief computation show that the action of wsw^{-1} on V depends only on α, not the choice of w and s. Hence we can define $t_\alpha := wsw^{-1}$. We also note that t_α acts on V as a reflection sending α to $-\alpha$, fixing pointwise the hyperplane $H_\alpha := \{\lambda \in V \mid B(\lambda, \alpha) = 0\}$.

Let T denote the set of reflections $t_\alpha, \alpha \in \Phi$. Then we get that:

$$T = \bigcup_{w \in W} wSw^{-1}$$

The correspondence between α and t_α is bijective. Now we note the following result:

Theorem [2]:

(R1) *If we have $\alpha, \beta \in \Phi$, such that $\beta = w(\alpha)$ for some $w \in W$, then $wt_\alpha w^{-1} = t_\beta$, i.e., $wt_\alpha w^{-1} = t_{w(\alpha)}$.*

(R2) *let $w \in W$, $\alpha \in \Phi^+$. Then $\ell(wt_\alpha) > \ell(w)$ if and only if $w(\alpha) > 0$.* □

Deletion and Exchange Condition

Here we outline a key fact about the nature of reduced expressions in W, which is an integral property of Coxeter groups.

Strong exchange condition: let $w = s_1 \ldots s_r$ (not necessarily reduced) for $s_i \in S$. Suppose a reflection $t \in T$ satisfies $\ell(wt) < \ell(w)$. Then there is an index i for which

$$wt = s_1 \ldots \hat{s}_i \ldots s_r$$

where $\hat{s}_i$ indicates omission of s_i. If the expression for w is reduced, then i is unique.

We obtain the following result as a corollary of the proof of the strong exchange condition:

Deletion condition: let $w = s_1 \ldots s_r$ for $s_i \in S$, such that $\ell(w) < r$ (i.e. the expression for w is not reduced). Then there exist indices $i < j$ for which

$$w = s_1 \ldots \hat{s}_i \ldots \hat{s}_j \ldots s_r$$

where a hat again indicates omission. From this we see that if $w = s_1 \ldots s_r$ for $s_i \in S$, then a reduced expression for w may be obtained by omitting an even number of specific s_i.

Cayley graphs

Cayley graphs provide a convenient way of encoding properties of a group in a geometric fashion. For a (finitely generated) group G and generating set S, we define the **Cayley graph** $Cay(G,S)$ to be the graph such that:

- Each element of G is assigned a vertex.
- Whenever $g \in G$ and $s \in S$, there is an edge from g to gs.

Under the assumption that S generates G, $Cay(G,S)$ is connected, directed and locally finite. By convention, if a generator $s \in S$ has order 2 in G, then we include a single undirected edge between g and gs.

For example, the Cayley graph for any given triangle group can be realised by taking the dual graph of its respective tiling:

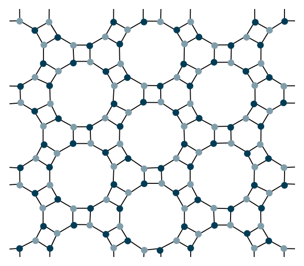

Figure 5. $\Delta(2,3,6)$ Cayley graph

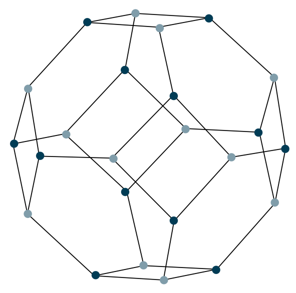

Figure 6. $\Delta(2,3,3)$ Cayley graph

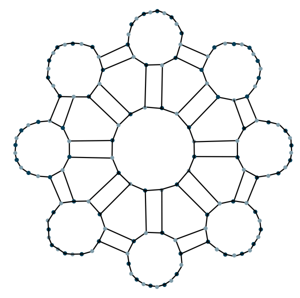

Figure 7. Portion of $\Delta(2,3,8)$ Cayley graph

Closely related to the Cayley graphs of Coxeter groups, is the **Davis complex** – an alternate geometric realisation of Coxeter groups that is CAT(0) for every Coxeter group [4]. Each Coxeter group acts on its Davis complex, in a way that allows us to hold on to the orthogonality of the associated reflections, but requires us to forgo linearity.

Contragredient action and the Tits cone

For any subset $I \subset S$, Define W_I to be the subgroup of W generated by all $s \in I$. All subgroups of W that can be obtained in this way are called **parabolic subgroups**. Importantly, W_I itself is a Coxeter group, relative to the generating set I.

We now aim to further investigate the action of parabolic subgroups with particular emphasises on the role of reflecting hyperplanes. To do so, we will consider how W acts on V^* by considering the contragredient action:

$$\sigma^* : W \to GL(V^*)$$

Throughout, we will denote the elements of V^* by $f, g, h, \ldots$, and following Humphreys, we will introduce the following notation: for $\lambda \in V$ and $f \in V^*$, denote $f(\lambda) = \langle f, \lambda \rangle$. Then the action of W on V^* is characterised by:

$$\langle w(f), \lambda \rangle = \langle f, w^{-1}(\lambda) \rangle$$

where we again use $w(\lambda)$ and $w(f)$ to denote $\sigma(w)(\lambda)$ and $\sigma^*(w)(f)$ respectively. For each $s \in S$, we define the hyperplane: $H'_s = \{f \in V^* \mid \langle f, \alpha_s \rangle = 0\}$ and the associated positive half space: $A_s = \{f \in V^* \mid \langle f, \alpha_s \rangle > 0\}$. Now Define

$$C = \bigcap_{s \in S} A_s \text{ and } D = \overline{C} = \bigcap_{s \in S} \overline{A_s}$$

The action of W and its parabolic subgroups W_I can now be analysed by partitioning D into subsets C_I defined as follows:

$$C_I = \left(\bigcap_{s \in I} H'_s \right) \cap \left(\bigcap_{s \notin I} A_s \right)$$

At the extremes, $C_\emptyset = C$ and $C_S = \{0\}$. Now define the **Tits cone**:

$$\mathscr{T} = \bigcup_{w \in W} w(D)$$

This is a W stable subset of V^* and is the union of all sets of the form $w(C_I)$ where $w \in W$ and $I \subset S$. We can now state the following results [2]:

Theorem:

(a) *W_I is the stabiliser in W of each point of C_I, and $\mathscr{T}$ is partitioned by all sets of the form $w(C_I)$ where $w \in W$ and $I \subset S$.*

(b) *D is the fundamental domain for the action of W on $\mathscr{T}$, i.e., the W-orbit of each point of $\mathscr{T}$ meets D at exactly one point.* □

The imaginary cone and its relationship with the Tits cone

Given a Coxeter system (W,S), a based root system as defined in [1] (Φ, Π) has the following properties:

- $\Pi \subset \Phi$ is positively independent
- $S = \{s_\alpha \mid \alpha \in \Pi\}$
- $\Phi = \{w(\alpha) \mid w \in W, \alpha \in \Pi\}$

Define the **Imaginary cone** $\mathscr{K}$ of a based root system (Φ, Π) as follows [3]:

$$\mathscr{K} = \bigcup_{w \in W} w(\mathscr{Y}) \subset V$$

where: $\mathscr{Y} = \{v \in cone(\Pi) \mid B(v, \alpha) \leq 0, \ \forall \alpha \in \Pi\}$

Theorem: *Under the assumption that B is non-singular, V is finite dimensional and Π is finite, the closures $\overline{\mathscr{K}}$ and $\overline{\mathscr{T}}$ of the imaginary cone $\mathscr{K}$ and the Tits cone $\mathscr{T}$ are dual cones . That is,*

$$\overline{\mathscr{K}^*} = \overline{\mathscr{T}}$$

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tjdavies@student.unimelb.edu.au