Symmetric polynomials

Shir Levav-Porat

Supervisor: Dr. Peter McNamara

Introduction

In our world, we see symmetries everywhere; butterfly wings, shell patterns, and even in Taylor Swift's face. We call a polynomial $f(x_1, ..., x_n)$ **symmetric** in n variables if $\pi(f) = f$ for all permutations $\pi \in S_n$. So, no matter how you swap the variables around, the polynomial remains unchanged. The set of all symmetric polynomials in n variables, $\mathbf{\Lambda(X_n)}$, is a vector space over $\mathbb{Q}$. When restricted to only the polynomials in which every term has a total degree of k, $\mathbf{\Lambda_k(X_n)}$ is finite dimensional.

Bases for $\Lambda_k(X_n)$

The **monomial** basis when $n \geq k$, $\{m_\lambda(X_n) \mid \lambda \vdash k\}$, are the polynomials formed by adding all the unique images of a monomial $\prod_{j=1} x_j^{\lambda_j}$ under the elements of S_n. There are other known basis elements;

Elementary	$\mathbf{e_a}(X_n) = \sum_{1\leq j_1<...<j_a\leq n} x_{j_1}...x_{j_a}$	$e_2(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3$
Homogenous	$\mathbf{h_a}(X_n) = \sum_{1\leq j_1\leq...\leq j_a\leq n} x_{j_1}...x_{j_a}$	$h_2(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2$
Power Sum	$\mathbf{p_a}(X_n) = \sum_{i=1}^n x_i^a$	$p_2(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$

with $a \in \mathbb{N}$. For a partition $\lambda = (\lambda_1, ..., \lambda_m)$, we define $e_\lambda = e_{\lambda_1}e_{\lambda_2}...e_{\lambda_m}$ and similarly for h_λ and p_λ.

Partitions and Tableaux

A partition $\lambda = (\lambda_1, ..., \lambda_m)$ of $k \in \mathbb{N}$ is a weakly decreasing sequence of non-negative integers such that $\sum_{i=1}^m \lambda_i = k$. We write $\lambda \vdash k$. The **Young** diagram of λ has λ_i boxes in the i^{th} row. For example, the corresponding tableau to the shape $\lambda = (4, 2)$ is

$$\begin{array}{|c|c|c|c|}\hline & & & \\ \hline & \\ \cline{1-2}\end{array}$$

We can fill these diagrams with a content $\mu = (\mu_1, ..., \mu_k)$ by placing each μ_i in a box. If the rows of the filled diagram are weakly increasing, and the columns strictly increasing, it is a **semi-standard** Young Tableau (SSYT). The reading word of a tableau is the word obtained by concatenating the rows left to right, starting from the shortest row.

Schur Polynomials

There's another basis for Λ_k - the **Schur** polynomials $s_\lambda(X_n)$. These can be defined by the contents of all SSYT of shape $\lambda \vdash k$ filled with integer elements of $\{1, ..., n\}$;

$$s_\lambda(X_n) = \sum_{T\in SSYT_n(\lambda)} x^T.$$

For example, with $\lambda = (3, 1)$ and $n = 2$:

$$\begin{array}{|c|c|c|}\hline 1&1&1\\ \hline 2\\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1&1&2\\ \hline 2\\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1&2&2\\ \hline 2\\ \cline{1-1}\end{array}$$

$$s_{(3,1)}(x_1, x_2) = x_1^3x_2 + x_1^2x_2^2 + x_1x_2^3$$

Robinson–Schensted–Knuth (RSK) Algorithm

The RSK correspondence is a **bijection** between all non-negative integer matrices A with a fixed amount n of non-zero entries and pairs (P, Q) of SSYT that have the same shape $\lambda \vdash n$, using a generalised permutation. This is a $2 \times n$ matrix, π, in which the top row is weakly increasing, and the bottom row is weakly increasing in sections with the same top entry. It is constructed by adding each pair (i, j) as a column to the matrix π exactly $(A)_{ij}$ times, following the rules above.

Insert the leftmost entries of both the of the bottom row and top row of π into separate tableaux, P and Q respectively. Moving left to right through the **bottom** row of π, take the new entry, e, and check the first row of P. If this row has no larger entries than e, append e to the end. If it does, say $j > e$ with j the leftmost entry satisfying this, replace j with e and move down to the next row, repeating the process with j and any other subsequent replaced entries until a new box is attached. Note where this new box is appended and add the corresponding top entry value above e to that same box in Q.

Example: RSK

We start by constructing the generalised permutation (GP) π from the matrix A:

$$A = \begin{bmatrix} 0&0&2\\ 1&3&0\\ 1&0&0 \end{bmatrix} \to \pi = \begin{pmatrix} 1&1&2&2&2&2&3\\ 3&3&1&2&2&2&1 \end{pmatrix}.$$

We can then apply the RSK algorithm, described above, with the steps being:

$$P \to \begin{array}{|c|}\hline 3\\ \hline\end{array} \qquad Q \to \begin{array}{|c|}\hline 1\\ \hline\end{array}$$

$$P \to \begin{array}{|c|c|}\hline 3&3\\ \hline\end{array} \qquad Q \to \begin{array}{|c|c|}\hline 1&1\\ \hline\end{array}$$

$$P \to \begin{array}{|c|c|}\hline 1&3\\ \hline 3\\ \cline{1-1}\end{array} \qquad Q \to \begin{array}{|c|c|}\hline 1&1\\ \hline 2\\ \cline{1-1}\end{array}$$

$$P \to \begin{array}{|c|c|}\hline 1&2\\ \hline 3&3\\ \hline\end{array} \qquad Q \to \begin{array}{|c|c|}\hline 1&1\\ \hline 2&2\\ \hline\end{array}$$

$$P \to \begin{array}{|c|c|c|}\hline 1&2&2\\ \hline 3&3\\ \cline{1-2}\end{array} \qquad Q \to \begin{array}{|c|c|c|}\hline 1&1&2\\ \hline 2&2\\ \cline{1-2}\end{array}$$

$$P \to \begin{array}{|c|c|c|c|}\hline 1&2&2&2\\ \hline 3&3\\ \cline{1-2}\end{array} \qquad Q \to \begin{array}{|c|c|c|c|}\hline 1&1&2&2\\ \hline 2&2\\ \cline{1-2}\end{array}$$

So finally we end up with

$$P = \begin{array}{|c|c|c|c|}\hline 1&1&2&2\\ \hline 2&3\\ \cline{1-2} 3\\ \cline{1-1}\end{array} \text{ and } Q = \begin{array}{|c|c|c|c|}\hline 1&1&2&2\\ \hline 2&2\\ \cline{1-2} 3\\ \cline{1-1}\end{array}$$

The GP π can be remade by removing entries in Q of highest value and rightmost placement, then sliding out entries in P by reversing the row insertion process, showing this is a bijection.

Hall inner product

Using the RSK correspondence, we can prove **Cauchy's identities**:

$$\sum_\lambda s_\lambda(X)s_\lambda(Y) = \prod_{i,j} \frac{1}{1 - x_iy_j}.$$

$$\sum_\lambda h_\lambda(X)m_\lambda(Y) = \prod_{i,j} \frac{1}{1 - x_iy_j}.$$

The Hall inner product is a unique inner product on Λ_k such that $\langle h_\lambda, m_\mu\rangle = \delta_{\lambda\mu}$. It can then be shown that the conditions

(i) $\langle u_\lambda\, , v_\mu\rangle = \delta_{\lambda\mu}$
(ii) $\sum_\lambda u_\lambda(X)v_\lambda(Y) = \prod_{i,j} \frac{1}{1-x_iy_j}$

are equivalent for bases $\{u_\lambda\}$, $\{v_\lambda\}$ of Λ_k. From Cauchy's formulas, we hence see that $\{h_\lambda\}$ and $\{m_\lambda\}$ are dual bases, and $\{s_\lambda\}$ is self-dual.

Littlewood-Richardson (LR) Rule

The rule for multiplying Schur functions is

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu.$$

There are multiple ways to calculate $c^\nu_{\lambda,\mu}$, the LR coefficient for partitions with $|\nu| = |\lambda|+|\mu|$.

Calculating LR coefficients

One way to understand $c^\nu_{\lambda,\mu}$ is by counting the number of SSYT of shape ν with the boxes λ removed, filled with a weight μ such that any tail section of each reading word contains at least as many k's as it does $k + 1$'s.

A different way to find $c^\nu_{\lambda,\mu}$ comes from the crystal graphs of SSYT, and their tensor products. We count the pairs (T_1, T_2) of SSYT of shapes λ and μ such that applying RSK to the reading word of T_1 concatenated with T_2 returns P of shape ν. P must satisfy the rule that all entries in rows j are j. Consider the product s_λ^2 for $\lambda = (4, 2, 1)$. To compute $c^\nu_{\lambda,\lambda}$ of a specific $\nu = (6, 5, 3)$, we can find all these pairs of reading words that map via RSK to

$$P = \begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&1&1\\ \hline 2&2&2&2&2\\ \cline{1-5} 3&3&3\\ \cline{1-3}\end{array}$$

We compute that there are 2 such reading word pairs; 3231122 3221111 and 3221123 3221111.

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