Quantum Error-Correcting Codes: Beyond Single-Mode Codes

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Introduction

In quantum computation and quantum information transmission, quantum bits (qubits) are susceptible to noise, which can alter their quantum states. Therefore, quantum error-correcting codes are essential for protecting quantum information against noise, while being carefully designed to avoid collapsing the original quantum state and thereby destroying the encoded information.

Motivation: 1-Mode Rotation Codes

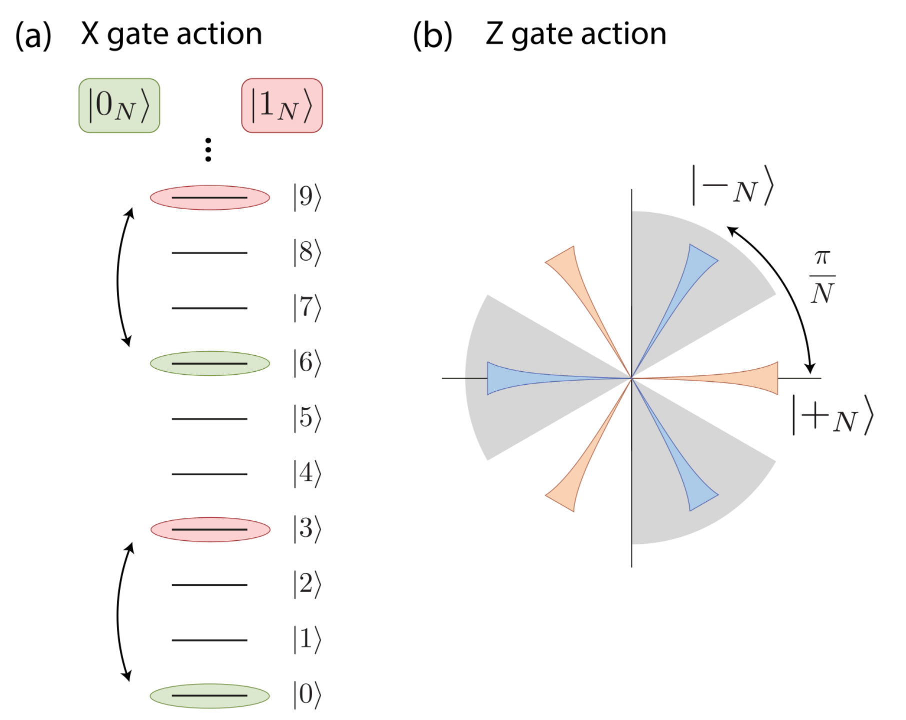

Figure 1: Action of gates on order-3 1-mode rotation code in Fock space (a) and phase space (b).

The 1-mode bosonic rotation codes protect quantum information using rotational symmetry in phase space and inspire multi-mode generalizations such as Tartan code and Chessboard code.

- The computational basis code words are

$$|0_N\rangle = \sum_{k=0}^{\infty} f_{2kN}\,|2kN\rangle \ ,$$
$$|1_N\rangle = \sum_{k=0}^{\infty} f_{(2k+1)N}\,|(2k+1)N\rangle$$

- The dual basis code words are

$$|\pm_N\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right)$$

- The Z and X stabilizers are

$$\hat{S}_Z = \exp\left[i\frac{2\pi}{N}\hat{n}\right] \text{ and } \hat{S}_X = \sum_{n=0}^{\infty} |n\rangle\langle n+2N|$$

- The logical $\bar{Z}_N$ and $\bar{X}_N$ gates are

$$\bar{Z}_N = \exp\left[i\frac{\pi}{N}\hat{n}\right] \text{ and } \bar{X}_N = \sum_{n=0}^{\infty} |n\rangle\langle n+N|$$

- The logical $\overline{\text{CZ}}$ gate for two modes with orders N_1, N_2 is

$$\overline{\mathrm{CZ}} = \exp\left[i\frac{\pi}{N_1 N_2}\,\hat{n}_1 \otimes \hat{n}_2\right]$$

- Errors are described by the basis

$$\hat{\mathbb{E}}_m(\theta) = e^{i\theta\hat{n}}\hat{E}_m \quad \text{where} \quad \hat{E}_m = \sum_{n=0}^{\infty} |n\rangle\langle n+m| \,.$$

  Here $\hat{E}_m$ lowers Fock states by m without $\sqrt{n}$ factors, modelling number-shift errors in rotation codes.

- Correctable errors for an ideal order-N 1-mode code satisfy

$$m \in \{0,\dots,N-1\} \quad \text{(number errors)}\ ,$$
$$\theta \in \left(-\frac{\pi}{2N},\frac{\pi}{2N}\right) \quad \text{(phase errors)}$$

- The number and rotation error distance of an ideal order-N 1-mode code are

$$d_n = N \quad \text{(number distance)}\ ,$$
$$d_\theta = \pi/N \quad \text{(phase distance)}$$

- The number-phase error trade-off is

$$d_n d_\theta = \pi$$

2-Mode Code: Tartan Codes

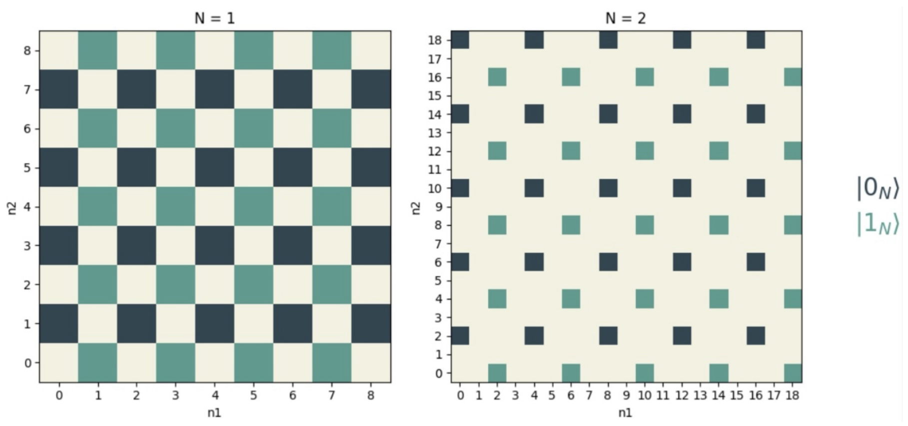

Figure 2: Tartan codes structure for 2-mode encoding with order N = 1 and 2. The light and dark green patterns encode the logical states $|1_N\rangle$ and $|0_N\rangle$ respectively (credit: M.Murali).

- The computational basis code words are

$$|0_N\rangle = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} g_{2k,2l+1}\,|2Nk\rangle \otimes |N(2l+1)\rangle \ ,$$
$$|1_N\rangle = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} g_{2k+1,2l}\,|N(2k+1)\rangle \otimes |2Nl\rangle$$

- The dual basis code words are defined similarly

$$|\pm_N\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right)$$

- The Z and X Stabilizers are

$$\hat{S}_{Z_1} = \exp\left[i\frac{2\pi}{N}\hat{n}_1\right] \text{ and } \hat{S}_{Z_2} = \exp\left[i\frac{2\pi}{N}\hat{n}_2\right] \ ,$$
$$\hat{S}_{X_1} = \hat{E}_{2N} \otimes \mathbb{1} \text{ and } \hat{S}_{X_2} = \mathbb{1} \otimes \hat{E}_{2N}$$

  where $\hat{n}_1 \equiv \hat{n} \otimes \mathbb{1}$ and $\hat{n}_2 \equiv \mathbb{1} \otimes \hat{n}$

- The logical $\bar{Z}_N$ and $\bar{X}_N$ gate are

$$\bar{Z}_N = \exp\left[i\frac{\pi}{N}\hat{n}_1\right] \text{ and } \bar{X}_N = \hat{E}_N \otimes \hat{E}_N$$

- The logical $\overline{Controlled - Z}$ gate is

$$\overline{\mathrm{CZ}} = \exp\left[i\frac{\pi}{N^2}\,\hat{n}_1 \otimes \hat{n}_2\right]$$

- Both modes can have errors $\hat{\mathbb{E}}_m(\theta)$
- Looking at the action of stabilizers on a generic error state
  $|\psi_E\rangle = \left(\hat{\mathbb{E}}_{m_1}(\theta_1) \otimes \hat{\mathbb{E}}_{m_2}(\theta_2)\right)\left(\alpha|0_N\rangle + \beta|1_N\rangle\right)$
  we see

$$\hat{S}_{Z_1}\hat{S}_{Z_2}\hat{S}_{X_1}\hat{S}_{X_2}|\psi_E\rangle$$
$$\approx \exp\left[\frac{2\pi i}{N}m_1 + \frac{2\pi i}{N}m_2 + i2N\theta_1 + i2N\theta_2\right]|\psi_E\rangle$$

  and the syndrome indicates the set of errors $\{\hat{\mathbb{E}}_{m_1}(\theta_1) \otimes \hat{\mathbb{E}}_{m_2}(\theta_2)\}$ are correctable for order-N Tartan code when

$$m_1, m_2 \in \{0,\dots,N-1\}\ ,$$
$$\theta_1,\theta_2 \in \left(-\frac{\pi}{2N},\frac{\pi}{2N}\right)$$

- The conjectured number and rotation error distance of an order-N Tartan code are

$$d_{m_1} = d_{m_2} = N\ ,$$
$$d_{\theta_1} = d_{\theta_2} = \pi/N$$

- The number-phase error trade-off is

$$\left(d_{m_1} + d_{m_2}\right)\left(d_{\theta_1} + d_{\theta_2}\right) = 4\pi$$

2-Mode Code: Chessboard Codes

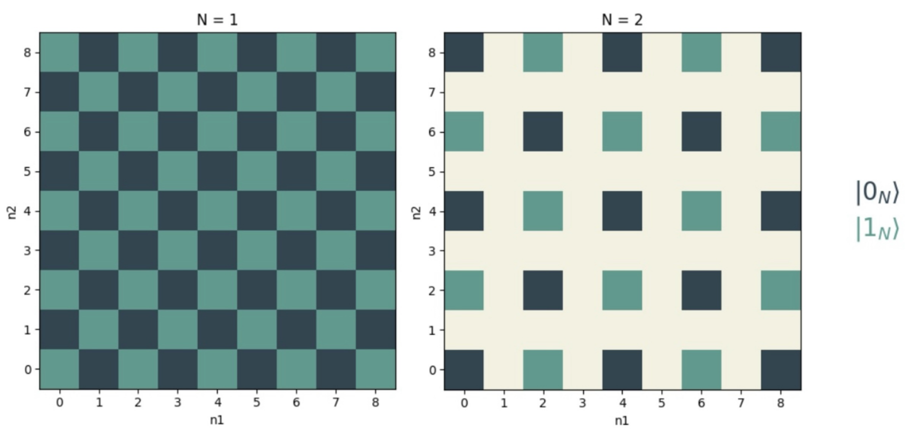

Figure 3: Chessboard codes structure for 2-mode encoding with order N = 1 and 2. The light and dark green patterns encode the logical states $|1_N\rangle$ and $|0_N\rangle$ respectively (credit: M.Murali).

- The computational basis code words are

$$|0_N\rangle = \sum_{\substack{k=0\\j=0}}^{\infty}\sum_{\substack{n_1=0\\n_2=0}}^{\infty} C_{n_1,n_2}\delta_{n_1,2jN}\delta_{n_1+n_2,2kN}\,|n_1\rangle \otimes |n_2\rangle \ ,$$
$$|1_N\rangle = \sum_{\substack{k=0\\j=0}}^{\infty}\sum_{\substack{n_1=0\\n_2=0}}^{\infty} C_{n_1,n_2}\delta_{n_1,2jN}\delta_{n_1+n_2,(2k+1)N}\,|n_1\rangle \otimes |n_2\rangle$$

- The dual basis code words are defined similarly

$$|\pm_N\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right)$$

- The Z and X Stabilizers are

$$\hat{S}_{Z_\pm} = \exp\left[i\frac{2\pi}{N}(\hat{n}_1 \pm \hat{n}_2)\right] \ ,$$
$$\hat{S}_{X_{p,q}} = \hat{E}_p \otimes \hat{E}_q$$

  where $p, q \in \mathbb{Z}_{\geqslant 0}$ and ($p+q = 2N$ or $|p-q| = 2N$)

- The logical $\bar{Z}_N$ and $\bar{X}_N$ gate are

$$\bar{Z}_{N_\pm} = \exp\left[i\frac{\pi}{N}(\hat{n}_1 \pm \hat{n}_2)\right] \ ,$$
$$\bar{X}_N = \hat{E}_p \otimes \hat{E}_q$$

  where $p, q \in \mathbb{Z}_{\geqslant 0}$ and ($p + q = N$ or $|p - q| = N$)

- The logical $\overline{Controlled - Z}$ gate is

$$\overline{CZ}_{N_1 N_{2\pm}} = \exp\left[i\frac{\pi}{N_1 N_2}(\hat{n}_1 \pm \hat{n}_2) \otimes (\hat{n}_1 \pm \hat{n}_2)\right]$$

- Looking at the action of stabilizers on a generic error state
  $|\psi_E\rangle = \left(\hat{\mathbb{E}}_{m_1}(\theta_1) \otimes \hat{\mathbb{E}}_{m_2}(\theta_2)\right)\left(\alpha|0_N\rangle + \beta|1_N\rangle\right)$
  we see

$$\hat{S}_{Z_+}\hat{S}_{Z_-}\hat{S}_{X_{2N,0}}\hat{S}_{X_{0,2N}}|\psi_E\rangle$$
$$\approx \exp\left[\frac{2\pi i}{N}(m_1 + m_2)\right]\exp\left[\frac{2\pi i}{N}(m_1 - m_2)\right]$$
$$\exp\left[i2N\theta_1 + i2N\theta_2\right]|\psi_E\rangle$$

  and the set of errors $\{\hat{\mathbb{E}}_{m_1}(\theta_1) \otimes \hat{\mathbb{E}}_{m_2}(\theta_2)\}$ are correctable for order-N Chessboard code when

$$m_1 + m_2 \in \{0,\dots,N-1\}\ ,$$
$$m_1 - m_2 \in \{0,\dots,N-1\}\ ,$$
$$\theta_1,\theta_2 \in \left(-\frac{\pi}{2N},\frac{\pi}{2N}\right)$$

- The conjectured number and rotation error distance of an order-N Chessboard code are

$$d_{m_1} = d_{m_2} = N\ ,$$
$$d_{\theta_1} = d_{\theta_2} = \pi/N$$

- The number-phase error trade-off is

$$\left(d_{m_1} + d_{m_2}\right)\left(d_{\theta_1} + d_{\theta_2}\right) = 4\pi$$

Discussion and Conclusion

The number and rotation error distances of both the Tartan and Chessboard codes are conjectured based on the syndromes of the corresponding error states. A natural direction for future work is to employ the Knill–Laflamme conditions to prove the conjectured distances.

Both two-mode codes enhance the number–phase error trade-off from π to 4π due to the additional mode, indicating that the logical qubits constructed from these codes are more tolerant to noise during quantum information transmission.

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References

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