QUESTION 1

The Best Die

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Abstract

A dice game is played. Two players take turns selecting one of five eight-sided dice, where player 2 is able to see the dice player 1 has chosen. After both players roll their dice, the player who rolls the highest value wins, while both players lose if they draw.

This game is an example of a complete information game. Both player 1 and player 2 play with the knowledge of the other's motive, and both know player 2 will have access to player 1's choice. Therefore the outcome of this game is more complex than a simple game of chance.

This report investigates which die is the best for either player to choose, and whether the addition of two faces to each die changes this strategy.

Because of the small and finite number of possible permutations, all arrangements of the two dice were modelled using a spreadsheet. Both the probability of winning in each scenario, and the probability of a draw were calculated. It was determined that player 1's ideal choice is dice 1 while player 2's ideal choices are dice 3 or 5 in response to player 1's choice of dice 1. A table of ideal responses to each of the five dice for player 2 was also identified.

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Problem Statement

For a 2 player game, there are 5 eight-sided dice on a table, all with different numerical values:
1. Die 1 - 4, 4, 4, 4, 4, 6, 6, 7.
2. Die 2 - 3, 3, 3, 3, 5, 5, 8, 8.
3. Die 3 - 1, 2, 2, 2, 8, 8, 8, 8.
4. Die 4 - 1, 1, 3, 6, 6, 6, 6, 7.
5. Die 5 - 0, 0, 5, 5, 5, 5, 6, 7.
Player 1 selects any one die. Player 2, who has seen the die that player 1 has chosen, can then select any one of the remaining 4 dice. Both players roll their dice. The winner is the player who rolls the highest number. If the numbers are the same, both players lose.
(a) What is the best die to choose?
(b) If we were to add two faces to each die, making them a ten sided die, with the two new faces taking the mean value of the eight sided die, would the result change?*

Assumptions

Below are explicit declarations of assumptions that can be inferred from the problem statement:

1. The dice are fair, meaning each face has the same probability of being rolled.
2. Each die can only be chosen once in one game.
3. Player 1 selects their die **with** the knowledge that Player 2 knows their selection.
4. Player 2 selects their die with the knowledge of point 3.
5. Player 1 will play optimally, and pick the die which will give them the greatest chance of a win.
6. As a draw will result in both players losing, both players will seek to prevent a permutation where a draw occurs. Hence, both players will consider drawing a "losing" strategy.

Analysis

Part A

First, it was identified that a total of only ${}^5P_2 = 20$ dice permutations exist, where ${}^nP_k = \dfrac{n!}{(n-k)!}$.

Hence, as the sample space is not unrealistically large, it was plausible to tackle this problem through the simulation of each dice permutation.

Let the die player 1 picks be dice X, and the die player 2 picks dice Y.

For each side of dice X and each side of dice Y, the probability of dice X being greater than dice Y (and thus, a win state) was collated into one win rate table. An excel spreadsheet was used to record each result (shown in the appendix).

A sample is shown below:

		Dice 1		Actual Dice #	1				
		7	6	6	4	4	4	4	4
Dice -1	8	-1	-1	-1	-1	-1	-1	-1	-1
	8	-1	-1	-1	-1	-1	-1	-1	-1
Actual Dice #	5	1	1	1	-1	-1	-1	-1	-1
2	5	1	1	1	-1	-1	-1	-1	-1
	3	1	1	1	1	1	1	1	1
	3	1	1	1	1	1	1	1	1
	3	1	1	1	1	1	1	1	1
	3	1	1	1	1	1	1	1	1

Total -1s	Total 0s	Total 1s	P(Dice -1 wins)	P(No wins)	P(Dice 1 wins)
26	0	38	13/32	#	19/32
			0.406	0	0.594

		Dice 1		Actual Dice #	3				
		8	8	8	8	2	2	2	1
Dice -1	8	0	0	0	0	-1	-1	-1	-1
	8	0	0	0	0	-1	-1	-1	-1
Actual Dice #	5	1	1	1	1	-1	-1	-1	-1
2	5	1	1	1	1	-1	-1	-1	-1
	3	1	1	1	1	-1	-1	-1	-1
	3	1	1	1	1	-1	-1	-1	-1
	3	1	1	1	1	-1	-1	-1	-1
	3	1	1	1	1	-1	-1	-1	-1

Total -1s	Total 0s	Total 1s	P(Dice -1 wins)	P(No wins)	P(Dice 1 wins)
32	8	24	1/2	1/8	3/8
			0.5	0.125	0.375

		Dice 1		Actual Dice #	4				
		7	6	6	6	6	3	1	1
Dice -1	8	-1	-1	-1	-1	-1	-1	-1	-1
	8	-1	-1	-1	-1	-1	-1	-1	-1
Actual Dice #	5	1	1	1	1	1	-1	-1	-1
2	5	1	1	1	1	1	-1	-1	-1
	3	1	1	1	1	1	0	-1	-1
	3	1	1	1	1	1	0	-1	-1
	3	1	1	1	1	1	0	-1	-1
	3	1	1	1	1	1	0	-1	-1

Total -1s	Total 0s	Total 1s	P(Dice -1 wins)	P(No wins)	P(Dice 1 wins)
30	4	30	15/32	1/16	15/32
			0.469	0.063	0.469

		Dice 1		Actual Dice #	5				
		7	6	5	5	5	5	0	0
Dice -1	8	-1	-1	-1	-1	-1	-1	-1	-1
	8	-1	-1	-1	-1	-1	-1	-1	-1
Actual Dice #	5	1	1	0	0	0	0	-1	-1
2	5	1	1	0	0	0	0	-1	-1
	3	1	1	1	1	1	1	-1	-1
	3	1	1	1	1	1	1	-1	-1
	3	1	1	1	1	1	1	-1	-1
	3	1	1	1	1	1	1	-1	-1

Total -1s	Total 0s	Total 1s	P(Dice -1 wins)	P(No wins)	P(Dice 1 wins)
28	8	11	7/16	1/8	11/64
			0.438	0.125	0.172

Above: All possible results of dice 2 combined with dice 1, 3, 4, 5.

If dice X beats dice Y, a value of 1 was assigned to the excel square. If the reverse was true, a value of -1 was assigned, whereas a value of 0 was assigned if the two dice drew. The number of resulting -1s, 0s, and 1s were summed, and the probability of each event was calculated, as:

$$\Pr(\text{outcome}) = \frac{\text{Number of favourable outcomes}}{\text{Number of total outcomes}}$$

For example, the probability of dice 2 defeating dice 3 is 1/2, or 50%. The probability of dice 2 losing to dice 1 is 3/8 or 37.5%. The probability of a draw is 12.5%.

In accordance with the problem statement, player 1 wins if dice X beats dice Y, but loses if dice X loses to dice Y **or** there is a draw. Thus, it is only in player 1's interest to consider their chances of **winning**, and not their chances of **not losing** (as this latter scenario includes the draw).

After simulating each of the 20 possible dice permutations, the resulting win rate summary table was created:

DICE X \ DICE Y	1	2	3	4	5
1	-	0.59	0.50	0.44	0.45
2	0.41	-	0.50	0.47	0.44
3	0.50	0.38	-	0.59	0.63
4	0.42	0.47	0.38	-	0.58
5	0.50	0.44	0.38	0.34	-

Above: Probability of Dice X winning against Dice Y, with regards to dice number

In addition, the following probability table of draw rates was created:

DICE X \ DICE Y	1	2	3	4	5
1	-	0.00	0.00	0.14	0.05
2	0.00	-	0.13	0.06	0.13
3	0.00	0.13	-	0.03	0.00
4	0.14	0.06	0.03	-	0.08
5	0.05	0.13	0.00	0.08	-

Above: Probability of a draw between Dice X and Dice Y, with regards to dice number

Without consideration to game theory, it can be seen that the die with the best possible winning odds for player 1 (dice X) is dice 3, assuming it gets to play against dice 5. However, this is a large assumption that disregards turn order.

As player 2 (dice Y) is able to choose their die second, they are given the choice to inspect player 1's choice of die. After observing player 1's dice choice, it is clear that player 2 will not give player 1 this optimal match up. Instead, player 2 will naturally choose the die with the best odds against player 1's die.

Additionally, as this is a game of perfect information, it is possible for both player 1 and player 2 to play optimally. Both players have full knowledge of all the faces on all dice, as well as what the best choice for the other player is to make in response to their own choice.

Since player 1 **knows** that player 2 will choose the die that gives them the best chance of winning in response to player 1, it is best for player 1 to then choose the die that leaves them with the highest chance of winning **if** player 2 maximises their own chance of winning.

The table below shows what die (or dice if multiple dice have the same odds of winning) player 2 should choose in response to player 1's choice of die for each die in the game. The odds of both players winning and the odds of a draw are also shown.

Player 1 Die	Player 2 Die	P(Player 1 Wins)	P(Draw)	P(Player 2 Wins)
1	3	32/64	0/64	32/64
1	5	29/64	3/64	32/64
2	1	26/64	0/64	38/64
3	1	32/64	0/64	32/64
3	2	24/64	8/64	32/64
4	3	24/64	2/32	38/64
5	3	24/64	0/64	40/64

It can be seen that player 1 has the best possible chance of winning if they were to choose dice 1 or dice 3 with odds of 32/64 or 50%. This would only occur if player 2 had no motivation in making player 1 lose and only cared about winning themselves. However, player 2 could also choose dice 5 or dice 2 in response to player 1 choosing dice 1 or dice 3 respectively.

Therefore, it is safer for player 1 to choose dice 1, as it has a 29/64 chance of winning if player 2 opts to minimise player 1's chance of winning, rather than the 24/64 chance of dice 3.

Part B

In Part B, each dice gained an additional two faces, each with the value of the mean of the first eight faces. The problem booklet specified the following: “if the mean value is a decimal, round down to the nearest integer.”

Interestingly, the floored means of all five dice was 4. Hence, an additional two sides with the value of 4 were added to all 5 dice, to form the following:

Dice 1	**7**	**6**	**6**	**4**	**4**	**4**	**4**	**4**	**4**	**4**
Dice 2	**8**	**8**	**5**	**5**	**4**	**4**	**3**	**3**	**3**	**3**
Dice 3	**8**	**8**	**8**	**8**	**4**	**4**	**2**	**2**	**2**	**1**
Dice 4	**7**	**6**	**6**	**6**	**6**	**4**	**4**	**3**	**1**	**1**
Dice 5	**7**	**6**	**5**	**5**	**5**	**5**	**4**	**4**	**0**	**0**

Similarly, the win rates of all individual dice were calculated via Excel spreadsheet.

DICE X \ DICE Y	**1**	**2**	**3**	**4**	**5**
1	-	0.52	0.46	0.40	0.39
2	0.34	-	0.48	0.44	0.40
3	0.40	0.40	-	0.52	0.52
4	0.37	0.48	0.42	-	0.51
5	0.44	0.48	0.44	0.40	-

Above: Probability of Dice X winning against Dice Y, with regards to dice number

DICE X \ DICE Y	1	2	3	4	5
1	-	0.14	0.14	0.23	0.17
2	0.14	-	0.12	0.08	0.12
3	0.14	0.12	-	0.06	0.04
4	0.23	0.08	0.06	-	0.09
5	0.17	0.12	0.04	0.09	-

Above: Probability of a draw between Dice X and Dice Y, with regards to dice number

The win rates were not that different. However, the draw rates of each dice were significantly different. This was to be expected, as the values added to each die all had the same values, and each addition made up ⅕ of each dice's total number of sides.

Another table was constructed, showing all the possible die that player 1 could choose and the optimal die for player 2 to choose in response:

Player 1 Die	**Player 2 Die**	**P(Player 1 Wins)**	**P(Draw)**	**P(Player 2 Wins)**
1	5	39/100	17/100	44/100
2	1	34/100	14/100	52/100
3	2	40/100	12/100	48/100
4	3	42/100	6/100	52/100
5	3	44/100	4/100	52/100

It can be seen clearly from the table that the best choice for player 1 is to choose dice 5. This gives them a 44/100, or 44% chance of winning if player 2 chooses dice 3 in response. If player 1 indeed chooses dice 5, this gives player 2 a 52/100 or 52% chance of winning, which is also the best possible chance of winning for them.

Interestingly, if player 1 plays suboptimally, player 2 likely has a lower chance of winning. This is due to the higher probability of a draw compared to part A, in which neither player wins.

Conclusion

Based on individual case analysis of the possible permutations and win rates, and assuming that both players play optimally, player 1's ideal dice choice in part A is dice 1 (29/64 or 32/64 chance of winning). On the other hand, player 2's ideal dice choices in part A are dice 3 and 5 in response to player 1's choice, giving them a 32/64 chance of winning.

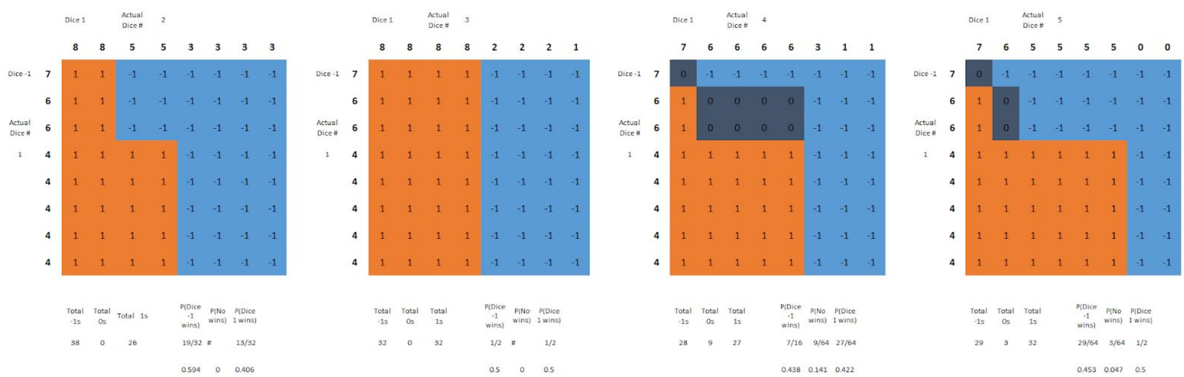
In part B, player 1 is instead most likely to win if they choose dice 5 (with a 44% chance of winning), compared to player 2's optimal choice of dice 3 (52% chance of winning) in response.

Appendix

Part A

Dice 1	7	6	6	4	4	4	4	4
Dice 2	8	8	5	5	3	3	3	3
Dice 3	8	8	8	8	2	2	2	1
Dice 4	7	6	6	6	6	3	1	1
Dice 5	7	6	5	5	5	5	0	0

Figure 0: List of dice



Dice 1 Actual Dice # 2

		8	8	5	5	3	3	3	3
Dice -1	7	1	1	-1	-1	-1	-1	-1	-1
	6	1	1	-1	-1	-1	-1	-1	-1
Actual Dice #	6	1	1	-1	-1	-1	-1	-1	-1
1	4	1	1	1	1	-1	-1	-1	-1
	4	1	1	1	1	-1	-1	-1	-1
	4	1	1	1	1	-1	-1	-1	-1
	4	1	1	1	1	-1	-1	-1	-1
	4	1	1	1	1	-1	-1	-1	-1

Total -1s	Total 0s	Total 1s	P(Dice -1 wins)	P(No wins)	P(Dice 1 wins)
38	0	26	19/32	#	13/32
			0.594	0	0.406

Dice 1 Actual Dice # 3

		8	8	8	8	2	2	2	1
Dice -1	7	1	1	1	1	-1	-1	-1	-1
	6	1	1	1	1	-1	-1	-1	-1
Actual Dice #	6	1	1	1	1	-1	-1	-1	-1
1	4	1	1	1	1	-1	-1	-1	-1
	4	1	1	1	1	-1	-1	-1	-1
	4	1	1	1	1	-1	-1	-1	-1
	4	1	1	1	1	-1	-1	-1	-1
	4	1	1	1	1	-1	-1	-1	-1

Total -1s	Total 0s	Total 1s	P(Dice -1 wins)	P(No wins)	P(Dice 1 wins)
32	0	32	1/2	#	1/2
			0.5	0	0.5

Dice 1 Actual Dice # 4

		7	6	6	6	6	3	1	1
Dice -1	7	0	-1	-1	-1	-1	-1	-1	-1
	6	1	0	0	0	0	-1	-1	-1
Actual Dice #	6	1	0	0	0	0	-1	-1	-1
1	4	1	1	1	1	1	-1	-1	-1
	4	1	1	1	1	1	-1	-1	-1
	4	1	1	1	1	1	-1	-1	-1
	4	1	1	1	1	1	-1	-1	-1
	4	1	1	1	1	1	-1	-1	-1

Total -1s	Total 0s	Total 1s	P(Dice -1 wins)	P(No wins)	P(Dice 1 wins)
28	9	27	7/16	9/64	27/64
			0.438	0.141	0.422

Dice 1 Actual Dice # 5

		7	6	5	5	5	5	0	0
Dice -1	7	0	-1	-1	-1	-1	-1	-1	-1
	6	1	0	-1	-1	-1	-1	-1	-1
Actual Dice #	6	1	0	-1	-1	-1	-1	-1	-1
1	4	1	1	1	1	1	1	-1	-1
	4	1	1	1	1	1	1	-1	-1
	4	1	1	1	1	1	1	-1	-1
	4	1	1	1	1	1	1	-1	-1
	4	1	1	1	1	1	1	-1	-1

Total -1s	Total 0s	Total 1s	P(Dice -1 wins)	P(No wins)	P(Dice 1 wins)
29	3	32	29/64	3/64	1/2
			0.453	0.047	0.5

Figure 1: Win rates of Dice 1

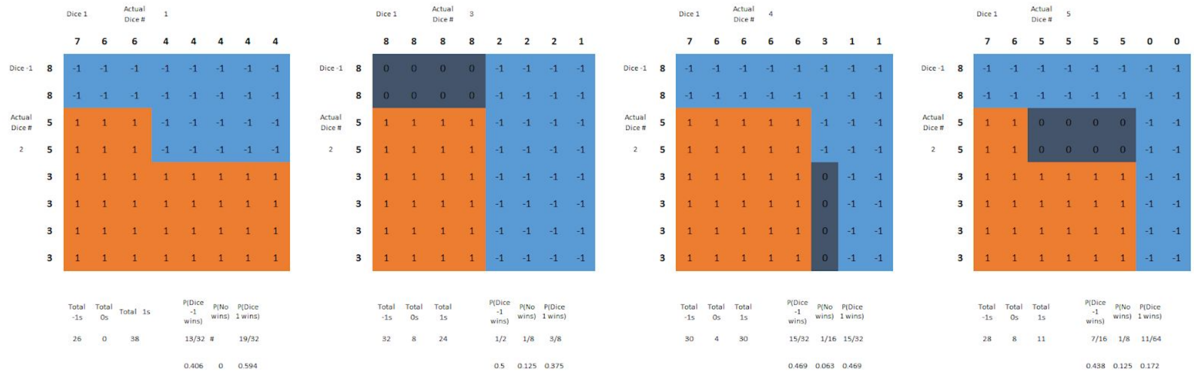


Figure 2: Win rates of Dice 2

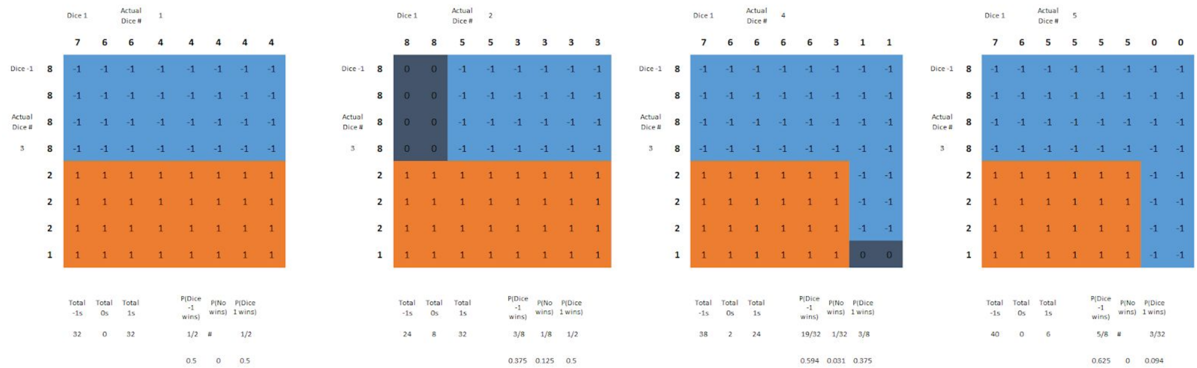


Figure 3: Win rates of dice 3

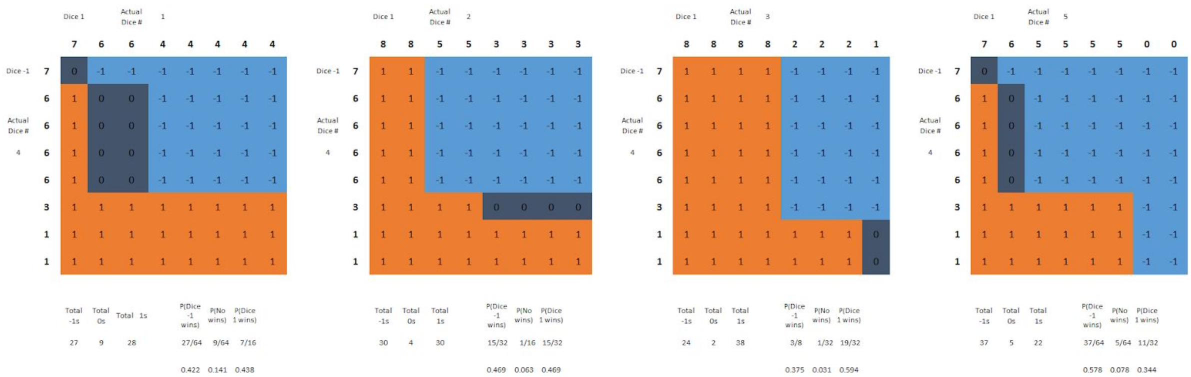


Figure 4: Win rates of dice 4

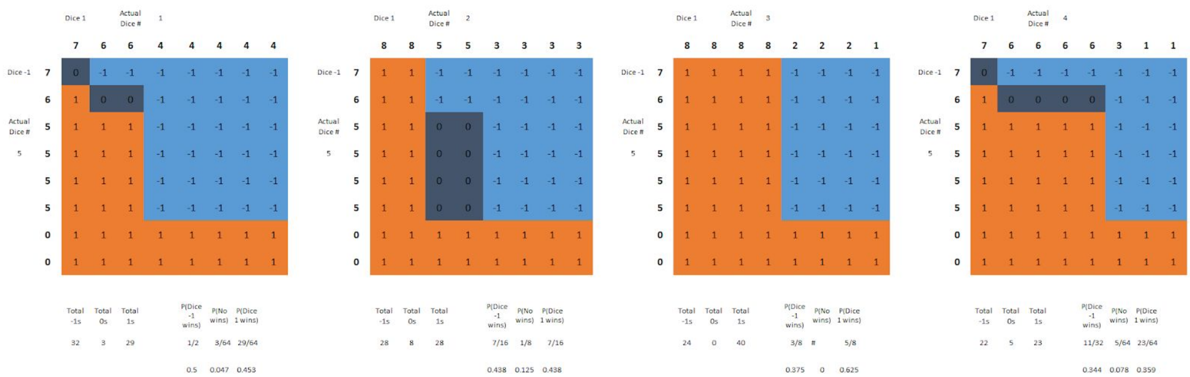


Figure 5: Win rates of dice 5

DICE Y

DICE X	1	2	3	4	5
1	-	0.59	0.50	0.44	0.45
2	0.41	-	0.50	0.47	0.44
3	0.50	0.38	-	0.59	0.63
4	0.42	0.47	0.38	-	0.58
5	0.50	0.44	0.38	0.34	-

Figure 6: Win rates of Dice X against Dice Y

DICE Y

DICE X	1	2	3	4	5
1	-	0.00	0.00	0.14	0.05
2	0.00	-	0.13	0.06	0.13
3	0.00	0.13	-	0.03	0.00
4	0.14	0.06	0.03	-	0.08
5	0.05	0.13	0.00	0.08	-

Figure 7: Draw rates of Dice X against Dice Y

Part B

Dice 1	**7**	**6**	**6**	**4**	**4**	**4**	**4**	**4**	**4**	**4**
Dice 2	**8**	**8**	**5**	**5**	**4**	**4**	**3**	**3**	**3**	**3**
Dice 3	**8**	**8**	**8**	**8**	**4**	**4**	**2**	**2**	**2**	**1**
Dice 4	**7**	**6**	**6**	**6**	**6**	**4**	**4**	**3**	**1**	**1**
Dice 5	**7**	**6**	**5**	**5**	**5**	**5**	**4**	**4**	**0**	**0**

Figure 0: List of dice

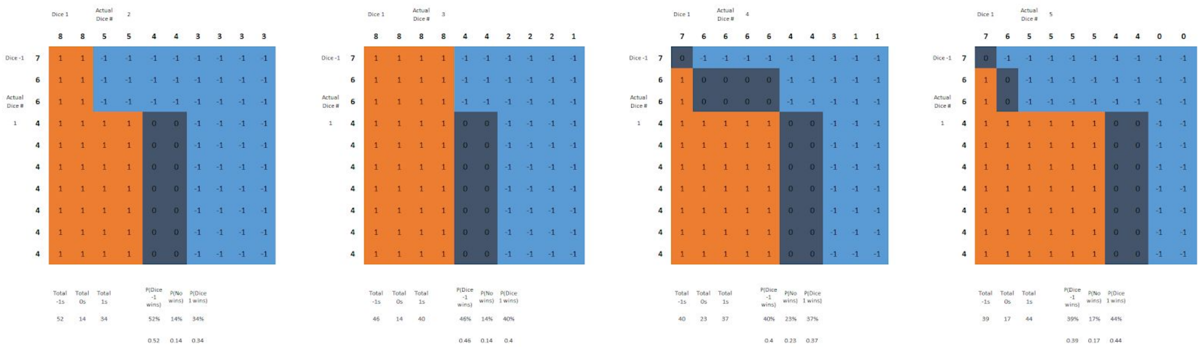


Figure 1: Win rates of dice 1

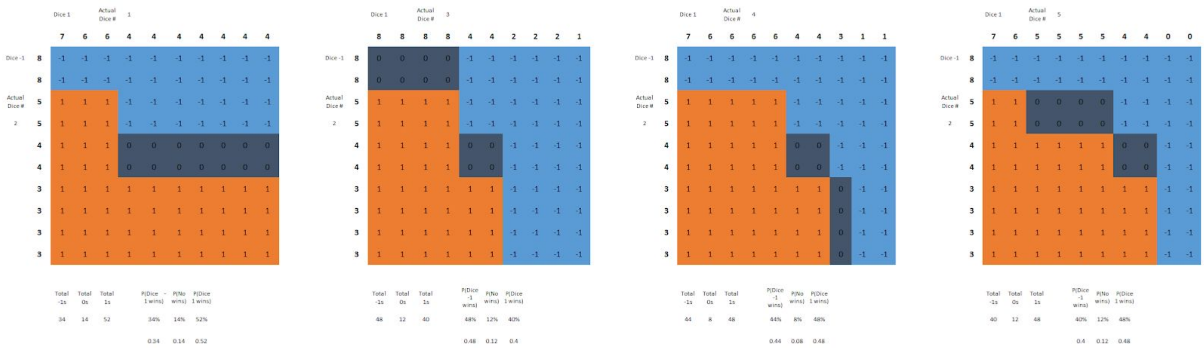


Figure 2: Win rates of dice 2

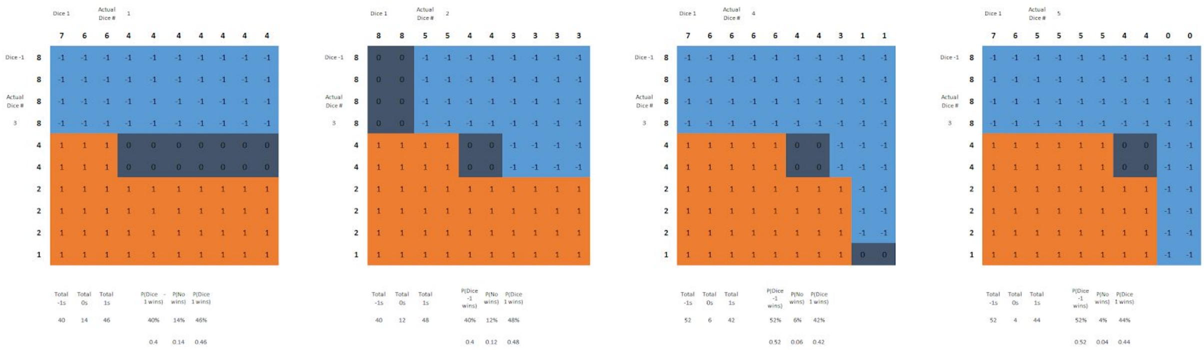


Figure 3: Win rates of dice 3

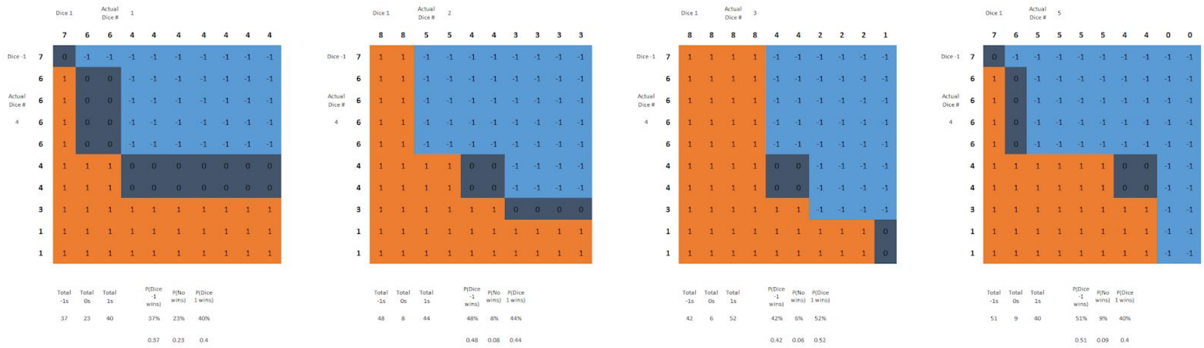


Figure 4: Win rates of dice 4

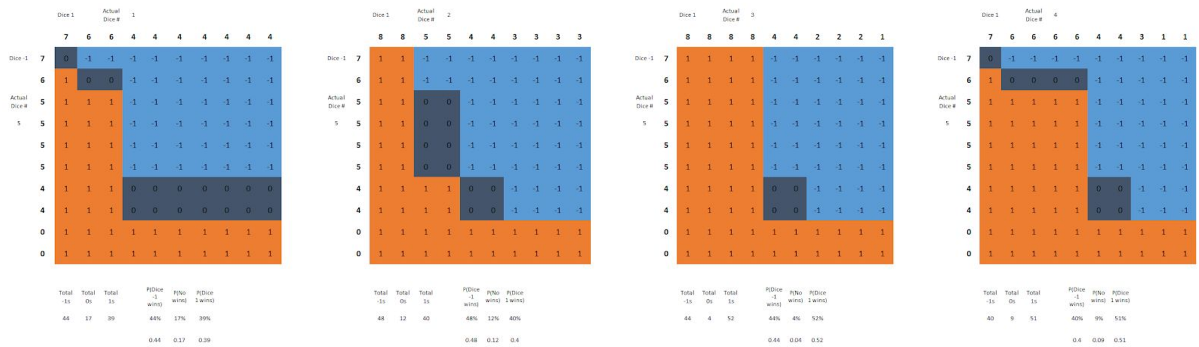


Figure 5: Win rates of dice 5

DICE X \ DICE Y	1	2	3	4	5
1	-	0.52	0.46	0.40	0.39
2	0.34	-	0.48	0.44	0.40
3	0.40	0.40	-	0.52	0.52
4	0.37	0.48	0.42	-	0.51
5	0.44	0.48	0.44	0.40	-

Figure 6: Win rates of Dice X against Dice Y

DICE X \ DICE Y	1	2	3	4	5
1	-	0.14	0.14	0.23	0.17
2	0.14	-	0.12	0.08	0.12
3	0.14	0.12	-	0.06	0.04
4	0.23	0.08	0.06	-	0.09
5	0.17	0.12	0.04	0.09	-

Figure 7: Draw rates of Dice X against Dice Y