Invariants of Knots

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Objectives

The aim of the project is to investigate various invariants of the knots.

How to tell two knots are the same?

What is a knot?
A **knot** is a smooth embedding of the circle S_1 into R^3 up to isotopy. i.e. a map $f : S_1 \to R^3$ without singularity such that any projection to R^2 is considered to be equivalent up to ambient isotopy.

"Equivalent" knots?
Informally, two knots are equivalent if one knot can be continuously deformed into another without breaking it.
Two knots K_1 and K_2 are **ambient isotopic** if there is an isotopy

$$f : R^3 \times [0, 1] \to R^3 \tag{1}$$

such that $f(K_1, 0) = K_1$ and $f(K_1, 1) = K_2$.

How tell if two knots are ambient isotopic?
Two knots are the equivalent if they differ by a composition of Reidemeister movement or planar isotopy.

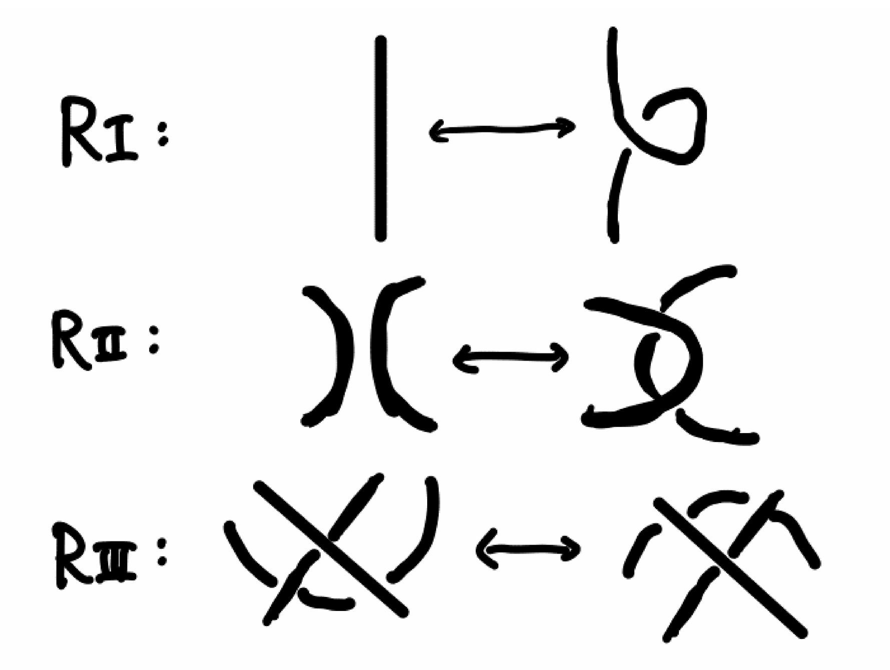

Figure 1:Reidemeister moves.

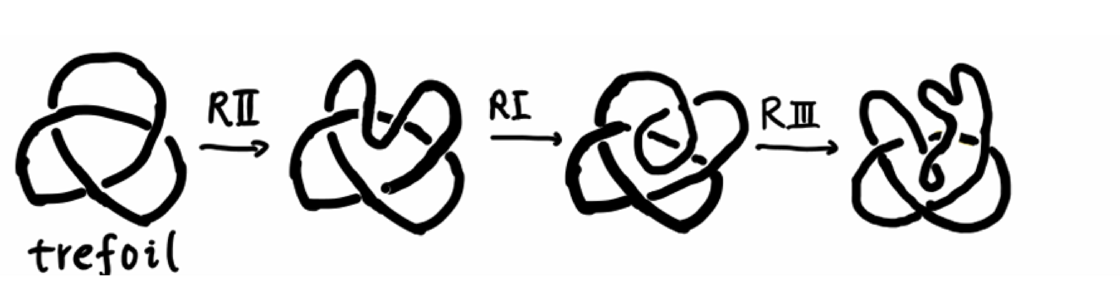

Figure 2:Ambient isotopic of trefoil.

How to distinguish knots?

Invariants Let K be the set of knots, S be any unspecific set.

$$V : K \to S$$

is an invariants, when $V(K_1) = V(K_2)$ if K_1 is ambient isotopic to K_2.

Tricolourability

A knot is tricolourable if each of the strands in the projection can be coloured by one of the three colours so that at each crossing, there will be one of the two scenarios:
1. The three strands are of three different colours,
2. The three strands have the same colour.
Tricolourability is an invariants for knots, so we can distinguish the unknot from the trefoil. But we cannot use tricolourability to distinguish trefoil and 7_4 knot.

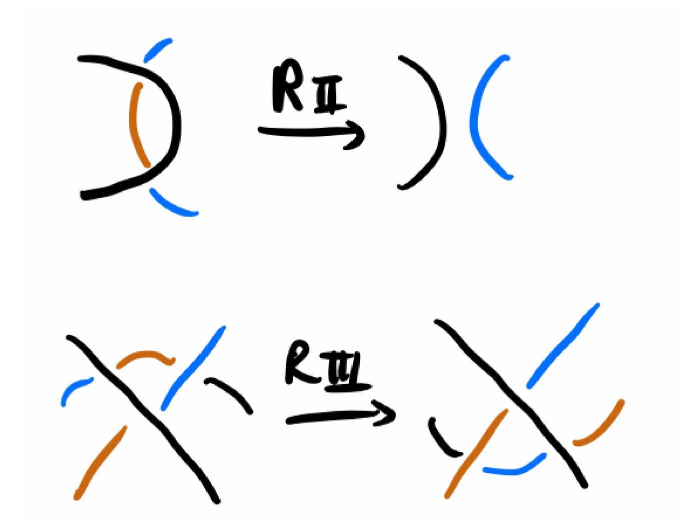

Figure 3:Tricolourability is invariants under Reidemeister moves.

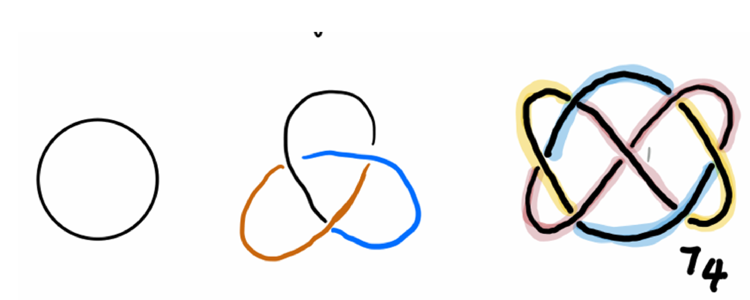

Figure 4:Tricolourability of the unknot, trefoil and 7_4

Bracket (Kauffman) Polynomial

The bracket polynomial is defined by the following three rules:

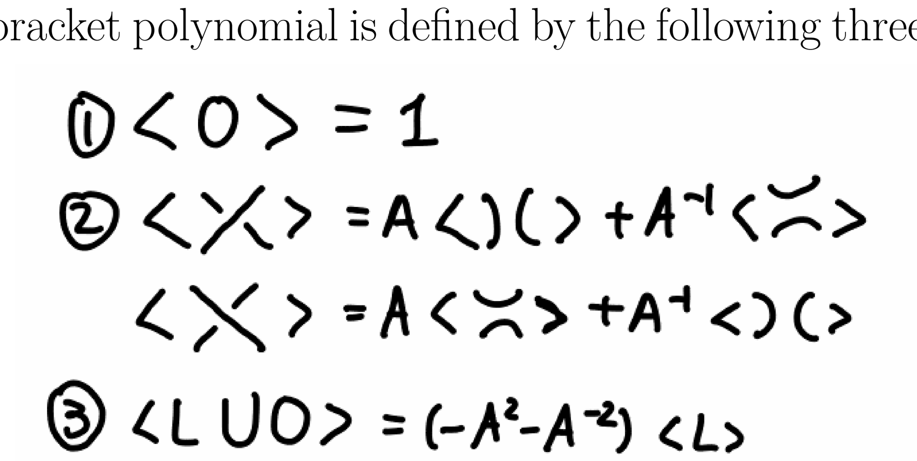

Figure 5:The three rules for bracket polynomial.

The bracket polynomial is invariant under RII and RIII, but not invariant under RI.

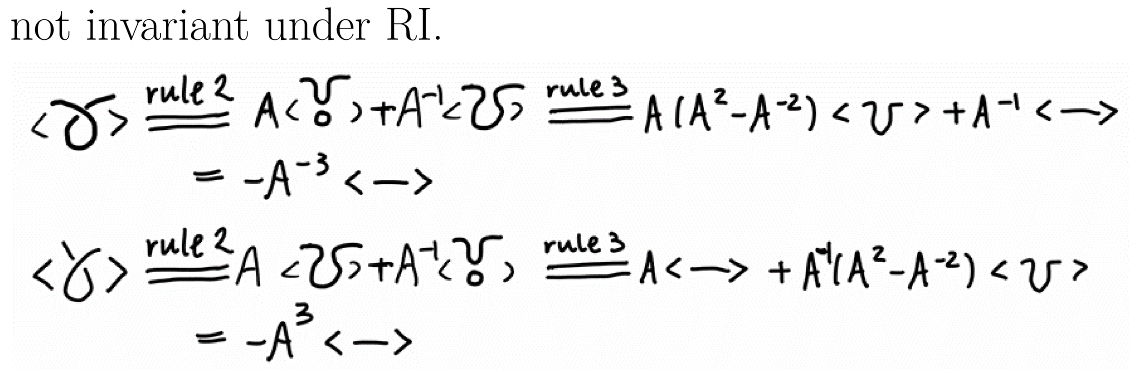

Figure 6:Bracket polynomial is not invariants under RI.

So the bracket polynomial is not an invariants, but we can build a polynomial invariants based on bracket polynomial.

X polynomials and Jones polynomials

Writhe saves the day!
• Pick an orientation of the knot projection, at each crossing, we have a either +1 or -1 defined below. (+1 if the strands can be lined up by rotating the under strand clockwise.) The writhe of the oriented knot projection is the sum of all the +1 or -1 at each crossing.

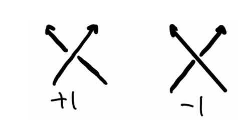

Figure 7:+1 and -1 crossing.

• It turned out that the writhe is an invariants under RII and RIII. RI changes the writhe by ± 1.

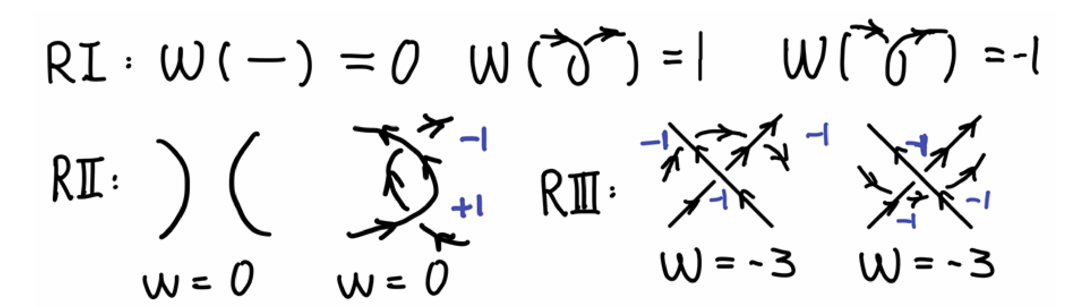

Figure 8:The writhe of knot projections under RI.

• We can use the writhe of the knot to define a new polynomial,

$$X(K) = (-A^3)^{-w(K)} < K >$$

so that it is invariants under RI.

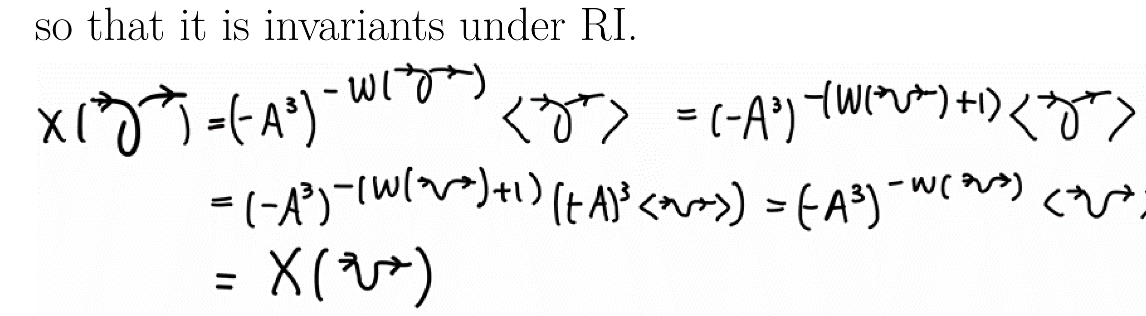

Figure 9:X polynomial is invariants under RI.

If we replace each A by $t^{-1/4}$, we obtain the Jones polynomials.

Bracket polynomial for the alternating knots

• **A-split and B-split**: Area A at a crossing is the area swiped over when rotating the overstrand counterclockwise.An **A-split** opens a channel between the two area A. B split is analogous to A split.

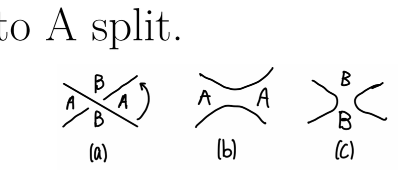

Figure 10:(a) Area A and Area B. (b) A split. (c) B split

• At each crossing, we make an A-split or B-split. The choice of how to to split all of n crossings in the knot projection is called a **state**. We will end up with various unknots.
• We denote $|S|$ as the number of unknots in the projection after splitting.Each time we have a A split, we multiply the resultant polynomial by A.
• So the bracket polynomial for the alternating knot K is

$$< K >= \sum_S A^{a(S)} A^{-b(S)} (-A^2 - A^{-2})^{|S|-1}$$

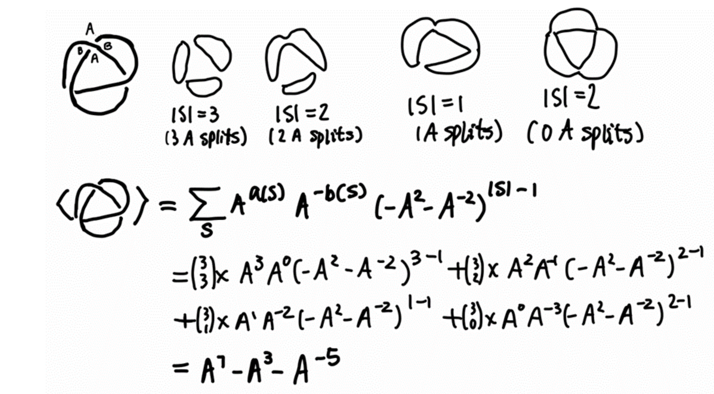

Figure 11:States of the trefoil knot and its bracket polynomial.

Conjecture

Now we can use bracket polynomial to prove a conjecture.

Conjecture: Two reduced alternating projections of the same knot have the same number of crossings.

Lemma: If K has a reduced alternating projection of n crossings, then span(<K>) = 4n.
• The span is the difference between the highest power and the lowest power that occurs in a polynomial. It is an invariant or knots.
• The highest power in the bracket polynomial is n+2(W-1) when we do all A-split at each crossing; The lowest power is -n-2(D-1). W(or D) denotes the number of white (or dark) area indicated below. Thus, the span<K> = 4n.

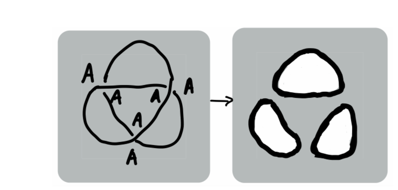

Figure 12:White and dark area.

Suppose P1, P2 are two reduced alternating projection of the knot K, and P1 has n crossings, then by the lemma, span(<P1>) = 4n.
Since span of the bracket polynomial is an invariants, then P2 also have n crossings, i.e. span(<P2>) = 4n = span(<P1>). Hence both alternating reduced projections of the same knot have the same number of crossings. □

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