Oriented graph colouring and reverse criticality

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Introduction

One method of defining a graph colouring is by way of graph homomorphism, a vertex mapping that preserves the existence of edges: $\chi(G) \leq t \iff G \to K_t$. The colour of each vertex is its image under this mapping.
Applying this definition to oriented graphs we obtain the following definition.

Definition 1 (Oriented Graph Colouring). A mapping from the vertices of G to a set of colours such that:

1. $c(u) \neq c(v)$ for every arc uv in G
2. $c(u) \neq c(y)$ for every pair of arcs uv and xy with $c(v) = c(x)$ [1]

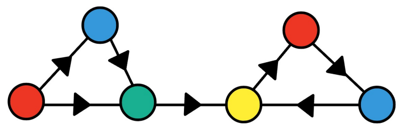

Figure 1: colouring of an oriented graph

Definition 2 (Oriented Chromatic Number). $\chi_o(G)$ is the smallest k such that there is an oriented colouring of G with k colours. [1]

From this concept, two key paths of enquiry occur;
1. how to determine $\chi_o(G)$ algorithmically, and
2. the existence of oriented graphs such that reversing any arc reduces $\chi_o(G)$.

Binary linear program to determine $\chi_o(G)$

$$\begin{array}{llr} \min & \sum X_i & \\ \text{s.t.} & \sum_{i \in C} X_{u,i} = 1 & u \in V \\ & X_{u,i} + X_{v,i} \leq X_i & uv \in A, i \in C \\ & X_{u,i} + X_{v,j} + X_{u',j} + X_{v',i} < 4 & uv, u'v' \in A \\ & X_i \in \{0,1\} & i \in C \\ & X_{u,i} \in \{0,1\} & u \in V, i \in C \end{array}$$

Theorem 1 The binary linear program above determines $\chi_o(G)$.

The constraints limit the scope of the binary linear program to valid oriented colourings. We have $X_i = 1$ when colour i is used. $X_{u,i} = 1$ when colour i is used on vertex u. Constraint 1 ensures each vertex has exactly 1 colour. Constraints 2 and 3 are the constraints from the definition of an oriented colouring.
The objective function minimises the number of colours used. By definition the smallest k such that there is a valid oriented colouring with k colours is $\chi_o(G)$.

Note: As there are no efficient algorithms to solve binary linear programs, this is not an efficient method for determining $\chi_o(G)$ of an oriented graph.

Reverse criticality

Definition 3 (Reverse criticality) An oriented graph G is reverse critical if the reversal of any arc in G decreases $\chi_o(G)$.

In looking for graphs that are reverse critical, we start by looking at cycles. The oriented chromatic number of orientations of cycles is fully classified in [1].

Theorem 2 An orientation of a cycle is reverse critical if and only if it is a directed cycle with $n \equiv 2 \pmod 3$ vertices.

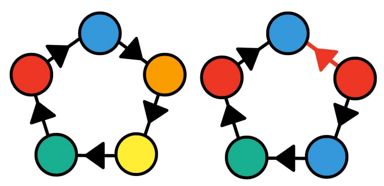

Figure 2: a reverse critical directed cycle

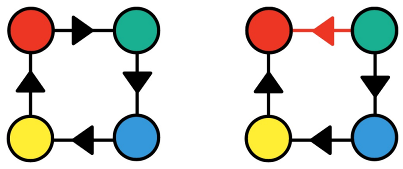

Figure 3: a directed cycle that is not reverse critical

Oriented cliques

Recall that a clique is a graph on n vertices for which $\chi(G) = n$. This motivates the following definition.

Definition 4 (Oriented Clique). An oriented graph G on n vertices is an oriented clique if $\chi_o(G) = n$.

Theorem 3 An oriented clique is an oriented graph where all pairs of vertices are connected by a path of length 1 or 2.

Definition 5 Let G and H be oriented graphs. Denote by $G \bowtie H$ the oriented graph constructed by connecting G and H via a universal vertex v, such that there is an arc from each vertex in $V(G)$ to v, and from v to each vertex in $V(H)$. [1]

Theorem 4 (Reverse critical clique construction). If G and H are reverse critical oriented cliques, then $G \bowtie H$ is a reverse critical oriented clique.

Iterating this construction, starting with $P_3 \bowtie P_3$, yields reverse critical oriented cliques with n vertices and $m = 2 + (n+1)\log_2(\frac{n+1}{4})$ arcs, where n is of the form $2^t - 1$.

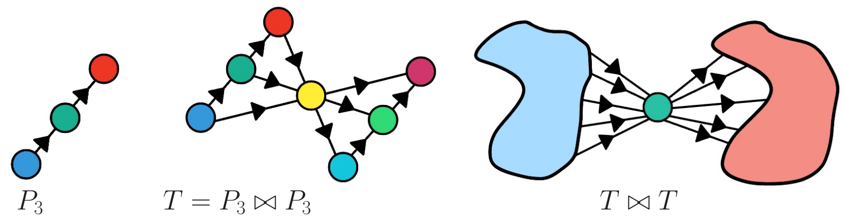

Figure 4: reverse critical clique construction

Iterating with $H = K_1$ in the construction yields reverse critical cliques with n vertices and $m = \frac{n^2-1}{4}$ arcs, where n is odd.

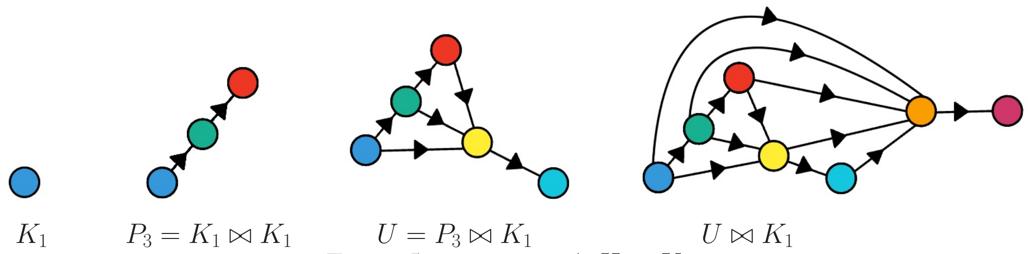

Figure 5: iterating with $H = K_1$

Definition 6 (Alternative reverse critical clique construction). For fixed k, begin with k vertices. For each sequence $\{+,-\}^k$, add a vertex where the arcs between the new vertex and the original k vertices are directed according to the entries in the sequence. This oriented graph has $n = 2^k + k$ vertices and $m = k2^k$ arcs.

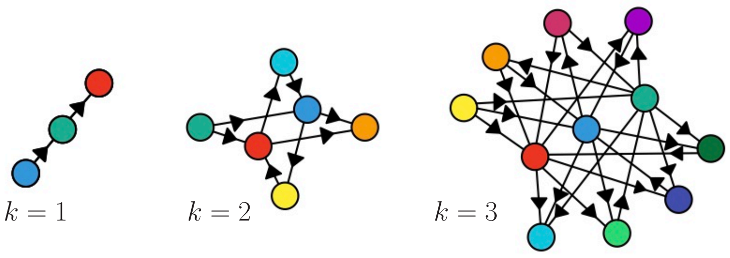

Figure 6: alternative reverse critical clique construction for $k = 1, 2, 3$

Minimising and maximising arcs in cliques

By computer search we find the reverse critical cliques with the least and most arcs for n from $1 - 9$. Denote these values as $s(n)$ and $l(n)$ respectively. **note:** there are no reverse critical cliques with 2 or 4 vertices.

n	$s(n)$	$l(n)$	Upper bound
1	0	0	0
2	—	—	1
3	2	2	2
4	—	—	4
5	5	6	6
6	8	9	10
7	10	12	14
8	13	16	18
9	15	21-24	24

Upper bound:
There are $\binom{n}{2}$ pairs of vertices in a reverse critical clique, m of these pairs are directly connected by an arc. Consider labelling the arcs in a reverse critical clique with all pairs of vertices that they are part of a directed 2-path between. Each arc must have at least 1 label to be critical and each pair of vertices will appear twice. Therefore, there are at least $\frac{m}{2}$ unique pairs labeled. From this, we obtain the inequality $m \leq \lfloor \frac{2}{3}\binom{n}{2} \rfloor$ providing an upper bound for the number of arcs in a reverse critical clique. As oriented graphs have integer number of arcs, we can take the floor of the upper bound.

Problems for further investigation

- Are the sequences $s(n)$ and $l(n)$ monotone increasing?
- Is there a counting argument that will provide an improved upper bound for $l(n)$?

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References

[1] Sopena, Eric. (2015). *Homomorphisms and colourings of oriented graphs: An updated survey*. Discrete Mathematics. 339. 10.1016/j.disc.2015.03.018.