Classical gauge theories: from Maxwell to Yang-Mills

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Introduction

Classical electromagnetism, when cast into the framework of special relativity, has five basic features

A Potential | Gauge Symmetry | Field Strength Tensor | Field Equations | An Invariant Action

These fundamental features of electromagnetism, in a generalised form, persist into the foundation of our modern theory of force fields, **Yang-Mills theory**. Through its adoption of Lie groups and Lie algebras, this theory takes the template of Maxwell theory and gives a diverse range of new physical field theories that continue to be essential to our understanding of fundamental interactions.

Note: This poster uses the Einstein summation convention. Summation over repeated indices is implied.

Electromagnetism

Special Relativity

In special relativity, everything happens in a flat 4-dimensional space-time called **Minkowski space**.

- It is equipped with the scalar product $X \cdot Y = X^\mu \eta_{\mu\nu} Y^\nu$
- The matrix $\eta_{\mu\nu}$ is called the **Minkowski metric**, with $\eta_{\mu\nu} = \text{diag}(1,-1,-1,-1)$ in Cartesian coordinates.
- Positions in Minkowski space-time are given by four-vectors $X^\mu = (ct, x, y, z)$, with indices $\mu = 0,1,2,3$ (0 being the time ordinate) and c being the speed of light.
- **covectors** have lowered indices, such as X_μ, and are related to their vector twin by the Minkowski metric $X_\mu = \eta_{\mu\nu} X^\nu$
- The **four derivative** ∂_μ is the gradient over space-time, with $\partial_\mu = (\partial_t, \vec{\nabla})$

A **Lorentz transformation** is a linear map on Minkowski space that preserves the scalar product.

Bringing Relativity to Electromagnetism

In the non-relativistic theory, we can define the scalar and vector potentials $\phi(\mathbf{x}, t)$ and $\mathbf{A}(\mathbf{x}, t)$ for a general configuration of the electric and magnetic (spatial) vector fields $\mathbf{E}$ and $\mathbf{B}$ to be such that

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

In four-dimensional space-time, we can package the scalar and vector potentials into a single new potential in the form of a four-vector,

$$A^\mu = \begin{pmatrix} \phi/c \\ \mathbf{A} \end{pmatrix}, A_\mu = \begin{pmatrix} \phi/c \\ -\mathbf{A} \end{pmatrix}$$

This is called the **four potential**. We can also define a one-form potential defined on an infinitesimal space tangent to the space-time manifold at all points, $A = A_\mu dx^\mu$

For a given configuration of $\mathbf{E}$ and $\mathbf{B}$ fields, A_μ is not unique. There is a **gauge symmetry** between potentials related by the gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \chi$$

Where χ is some arbitrary (differentiable) scalar function. Potentials related by gauge transformations produce identical physical $\mathbf{E}$ and $\mathbf{B}$ fields.

The Electromagnetic Field

We can also combine the electric and magnetic vector fields into a single tensor field over space-time.

We write the combined *electromagnetic field* as either a tensor $F^{\mu\nu}$ (a 4x4 antisymmetric matrix) or as a two-from F on 4-dimensional space-time.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = dA$$

The two-form has 6 degrees of freedom, one for each component of the electric and magnetic fields. We can also define the **Hodge dual**

$$\star F_{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

The Relativistic Maxwell's Equations

Maxwell's equations in a vacuum can be simply described in the relativistic framework by two linear partial differential equations

$$\partial_\mu F^{\mu\nu} = 0, \qquad \partial_\mu \star F^{\mu\nu} = 0$$

The Invariant Action

Another way to specify the behaviour of the electromagnetic field is using the **principle of stationary (or least) action**: For a given functional $S(A_\mu)$ called the *action*, the 'true' potential field is the one that corresponds to a stationary point in the action.

The action that correctly recovers Maxwell's equations is:

$$S = -\frac{1}{4\mu_0} \int_{\mathbb{R}^4} d^4x F_{\mu\nu} F^{\mu\nu}$$

Yang-Mills Theory

Lie Groups and Lie Algebras

At the heart of extending gauge theory beyond the realm of Maxwell are the concepts of Lie groups and their Lie algebras. Every Yang-Mills theory is founded upon a *Lie group*. Electromagnetism is itself Yang-Mills theory, based on the Abelian Lie group $U(1)$.

Lie groups and Lie algebras are intimately related structures.

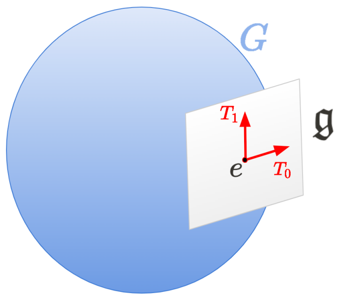

Figure 1. The manifold structure of a Lie group G with its Lie algebra $\mathfrak{g}$ given as the tangent space at the identity.

A **Lie group** is a group G which is also a manifold, meaning it can be described as a higher-dimensional surface that permits a *smooth parametrisation*.

A **Lie algebra** $\mathfrak{g}$ associated with any Lie group G is an infinitesimal vector space tangent to the identity of G.

- Its basis vectors X_a are called the *infinitesimal generators* of G.
- Formally, for Lie groups with matrix representations, we define $\mathfrak{g}$ as the space of $X \in G$ such that $e^{itX} \in G$ for all $t \in \mathbb{R}$
- The Lie algebra has a bilinear operation $[\cdot,\cdot]$ called the **Lie Bracket** that obeys the Jacobi identity. For matrix Lie groups, $[A, B] = AB - BA$

A standard choice of basis for the Lie algebra are the generators T^a such that $\text{Tr}T^aT^b = \frac{1}{2}\delta^{ab}$.

Example: the standard generators of the Lie group $SU(2)$ are the *Pauli matrices*,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These span the Lie algebra $\mathfrak{su}(2)$, the vector space of traceless Hermitian matrices.

Generalising Fields with Lie Algebras

As in electromagnetism, we can define a gauge potential A_μ:

- Each element belongs to the Lie algebra $A_\mu \in \mathfrak{g}$.
- With the generator basis T^a, $A_\mu = A^a_\mu T^a$
- Can also define a potential one-form $A = iA_\mu dx^\mu$

The force fields of Yang Mills theory can be described as either a tensor field $F^{\mu\nu}$ called the **field strength tensor** or as a **curvature** two-form F

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu], \qquad F = dA + A \wedge A$$

These objects are again Lie-Algebra valued and can be expanded as $F^{\mu\nu} = F^{a,\mu\nu} T^a$

The Yang-Mills Equations

For a Lie algebra-valued object ϕ, the **covariant derivative** with connection A_μ is $\mathcal{D}_\mu \phi = \partial_\mu \phi - i[A_\mu, \phi]$. The covariant derivative allows us extends Maxwell's equations to the non-commutative case: the **Yang-Mills equations**

$$\mathcal{D}_\mu F^{\mu\nu} = 0, \qquad \mathcal{D}_\mu \star F^{\mu\nu} = 0$$

Note these equations are **not linear** $\to$ self-interaction.

The Invariant Action

The Lorentz-invariant action that recovers the Yang-Mills equations is

$$S = -\frac{1}{4g^2} \int d^4x F^a_{\mu\nu} F^{a,\mu\nu} = -\frac{1}{2g^2} \int d^4x \text{Tr} F_\mu F^{\mu\nu}$$

Where g is the **coupling constant**.

The gauge symmetries under which this action is invariant come from the **gauge group**, the group of fields in space-time valued with $\Omega(x)$ from the underlying Lie group G. It acts on the gauge fields as

$$A_\mu \to \Omega A_\mu \Omega^{-1} - i\Omega \partial_\mu \Omega^{-1}, \quad F^{\mu\nu} \to \Omega F^{\mu\nu} \Omega^{-1}$$

References

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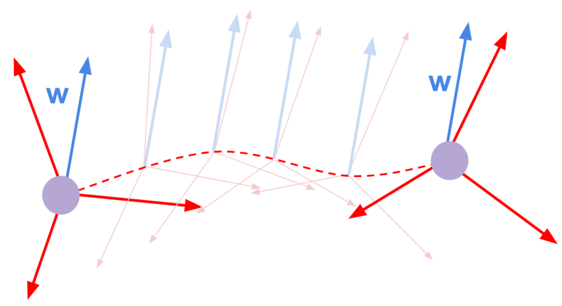

Figure 2. Yang-Mills theory also allow us to understand the potential A_μ as a connection relating internal degrees of freedom w in matter at different points in space-time when we allow for a simple gauge symmetry under $w \to \Omega(x) w$