Implementing symmetries by topological defects

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Motivations and background

Consider two types of topological objects in $\mathbb{R}^3$ — line and surface defects. These have a physical role in which they impose symmetries. Line defects are one-dimensional curves labeled with a group element from a group G, denoted L_g, $g \in G$. Similarly, surface defects are two-dimensional surfaces with an attached group element from a different group H, denoted D_σ, where $\sigma \in H$. These can interact both with themselves, through fusion, and with each other, wherein a line defect passes through a surface defect whereby the surface acts on the line defect. Surface fusion and line-surface interaction are illustrated below through figures 1 and 2.

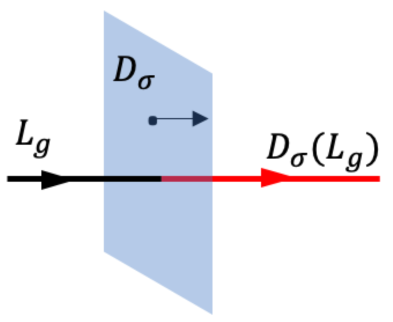

Figure 1. Line-surface interaction

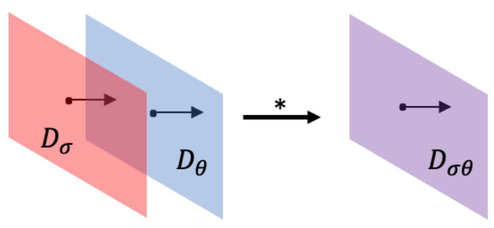

Figure 2. Surface fusion

For the purpose of this poster, only boundless surface and line defects will be considered. In this poster, this scenario shall be looked into; the consistency conditions will be investigated and the associated algebraic structure worked out.

Defects as groups

A Simple Case – Line and Surface Defects in Isolation

Given a set of line defects, between them one can define an operation of fusion ($*$) such that

$$L_g * L_h = L_{gh}.$$

With this there is a group. Associativity is inherited from G, and inverses are determined as follows: $(L_g)^{-1} = L_{g^{-1}}$. As the line defects are topological curves in $\mathbb{R}^3$ space, they can move around each-other without intersection, resulting in the order of fusion between two lines losing meaning. Subsequently, the group of line defects is abelian. Similarly, there is a group formed by surface defects and their respective fusion operation. This different group will not necessarily be abelian as $\mathbb{R}^3$ does not provide enough space to move two surface defects past eachother.

Surface and line defect interactions

With regard to line and surface defect interactions, we shall consider only line defects passing through surface defects at an individual point. When a line defect passes through a surface defect, we would like it to remain a line defect in the original group of line defects. In this sense, we can consider a line defect mapping to a line defect under surface defect crossing. As we are developing our defect structure via groups, this leads to surface defects behaving as automorphisms of the group of line defects. That is, $D_h \in \mathrm{Aut}(\text{line defect group})$.

Orientation of defects

In order to give meaning to the order of fusion (i.e., which surface or line defect is on which side of the operator), one must impose an orientation to the objects.

Surface orientation

The topological surfaces in consideration have normal vectors. Say that two surface defects can fuse if their normals are pointed in the same direction. The surface defect on the left of the fusion operation will be the surface whose normal is pointing into the other (Figure 2). If two surface defects have opposing directions, then this is equivalent to taking the inverse of one of the surfaces and then fusing. That is, to reverse orientation one can take the inverse of the surface defect (Figure 3).

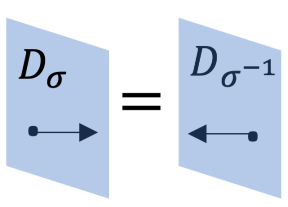

Figure 3. Surface inverse orientation

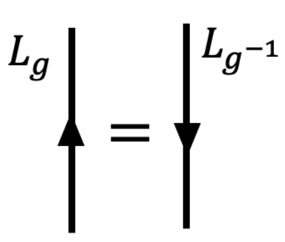

Figure 4. Line inverse orientation

Line orientation

While line defects are abelian and can move around each-other in $\mathbb{R}^3$, in order for the surface defects to act on them in a well-defined manner the line defects must also have orientation. The line will be equipped with an arrow as its orientation. From this, a line defect can be acted on by a surface defect when the surface defect's normal vector is in the same direction as the line's arrow (see Figure 4). A line defect's inverse will have the opposite orientation.

Computing the group of defect surfaces

The group of line defects is abelian. All finitely generated abelian groups are isomorphic to a direct sum of primary cyclic groups (cyclic groups with orders of a power of a prime) and powers of $\mathbb{Z}$. As such, automorphisms of cyclic groups will first be considered.

For the function $\Phi : G \to G$ to be an automorphism, it must fulfill the homomorphism requirement of

$$\Phi(gh) = \Phi(g) \circ \Phi(h), \forall g, h \in G.$$

Inner automorphisms of a group G are automorphisms of the form $\Phi_h : g \mapsto hgh^{-1}, h \in G$. For an abelian group G, these are the identity function. Hence, only outer automorphisms are nontrivial.

For now, let $G = \mathbb{Z}_m$. Recall that an element $n \in \mathbb{Z}_m$ can be expressed as $n = g^n$ by cyclic construction. Consequently, automorphisms satisfy

$$\Phi(n) = \Phi(g)^n.$$

From this, each automorphism can be characterised by where it maps the generator. Denote $\Phi_\sigma(g) = \sigma$ where g is the generator. Automorphisms must also be bijective. This is equivalent to $\mathbb{Z}_m$ being generated by σ. This holds if and only if m and σ are coprime. Thus, for $\mathbb{Z}_m$, it has the automorphisms

$$\{\Phi_\sigma : h \mapsto \sigma^h | \gcd(m, \sigma) = 1\}.$$

For $\mathbb{Z}$, if the integers are considered the additive group $\mathbb{Z}_N$ as N extends to infinity, then the only automorphisms are determined to be Φ_1 and Φ_{-1} (if we consider Φ to be $\{\Phi_\sigma : g \mapsto g * \sigma | \gcd(m, \sigma) = 1\}$).

To extend this to finitely generated abelian groups of distinct factors, one can take the direct sum of the automorphisms from each of its respective factors. This results in the automorphisms of that composite group being combinations of the automorphisms of its factors. That is, if $G = A \oplus B$ where A and B are distinct primary cyclic groups, then the automorphisms of G are given by $\{\Phi_\sigma \circ \Psi_\theta | \Phi_\sigma \in \mathrm{Aut}(A), \Psi_\theta \in \mathrm{Aut}(B)\}$. So for a group of line defects A of the form

$$A = \mathbb{Z} \oplus \mathbb{Z}_{N_1} \oplus \ldots \oplus \mathbb{Z}_{N_m}$$

where N_i are distinct powers of primes, the set of surface defects B can be described by

$$\begin{aligned} B &= \mathrm{Aut}(\mathbb{Z}) \oplus \mathrm{Aut}(\mathbb{Z}_{N_1}) \oplus \ldots \oplus \mathrm{Aut}(\mathbb{Z}_{N_m}) \\ &= \{\Phi_{(a,b_1,\ldots,b_m)} \mid a \in \{-1, 1\}, \gcd(b_j, N_j) = 1\} \end{aligned}$$

where Φ_σ retains its mapping of the generator to σ. To extend this to not necessarily distinct group factors, we must also consider automorphisms that permute indistinct factors. For example, if $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ then $\Psi : (a, b) \mapsto (b, a)$ is also an outer-automorphism of G. So for a group of line defects G isomorphic to a finitely generated abelian group, the group of surface defects is all of its automorphisms, which is the group of all compositions of the automorphisms of the cyclic factors of G and the permutations of its identitical factors.

Thus the set of surface defects has a group structure, with $D_\sigma^{-1} = D_{\sigma^{-1}}$, associativity inherited from its bijective nature and the identity given by D_e. This group is not generally abelian, as it is constructed by permutation of the automorphism compositions.

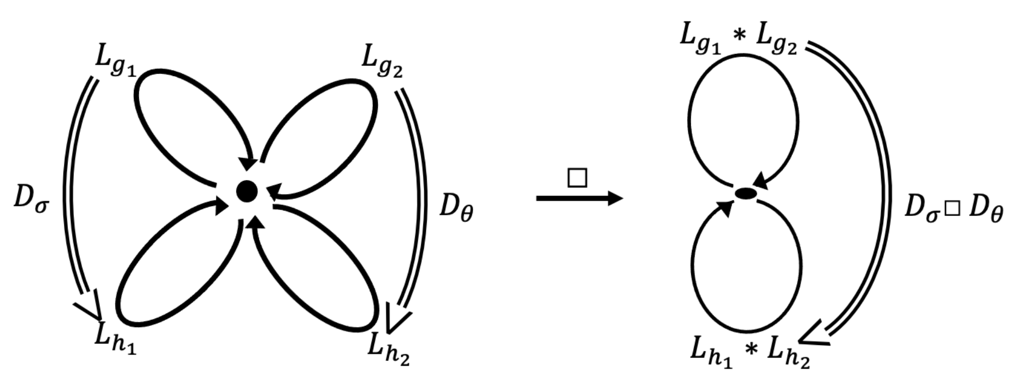

Figure 5. 2-category horizontal composition

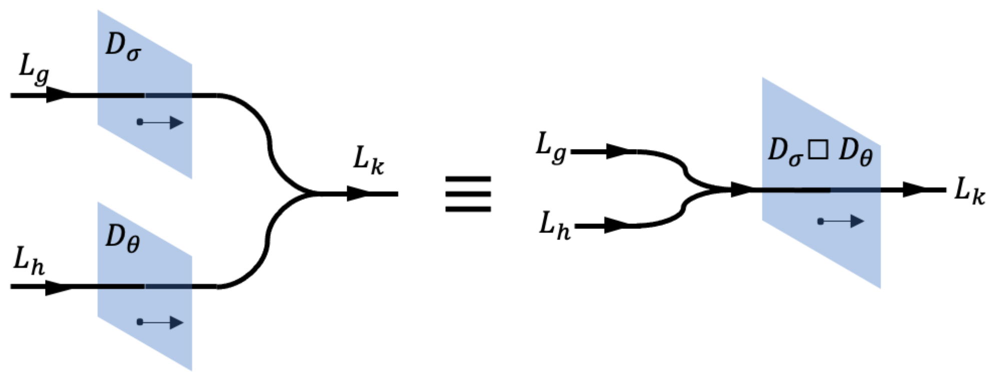

Figure 6. Horizontal composition physical interpretation

Imposed consistency conditions

By considering line and surface defects as groups in and of themselves, associativity holds with their own fusions. That is,

$$(L_g * L_h) * L_k = L_g * (L_h * L_k)$$

and

$$(D_\sigma \circ D_\theta) \circ D_\phi = D_\sigma \circ (D_\theta \circ D_\phi)$$

or equivalently,

$$L_{gh} * L_k = L_g * L_{hk} \quad \text{and} \quad D_{\sigma\theta} \circ D_\phi = D_\sigma \circ D_{\theta\phi}.$$

This has a physical effect, with the outputs of the fusion orders as in figure 7 being equivalent.

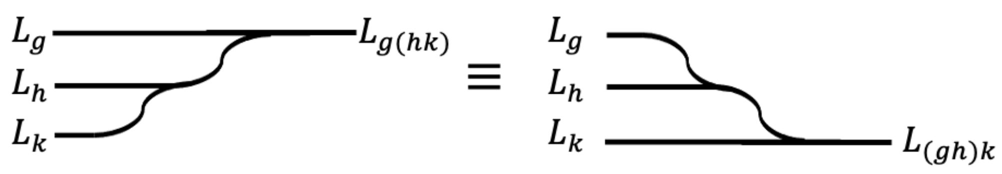

Figure 7. Line fusion associativity

From the surface defects acting as line defect automorphisms, it also follows that

$$D_\sigma(L_g * L_h) = D_\sigma(L_g) \circ D_\sigma(L_h).$$

It is natural to now wonder what properties the last combination of three defects have — two surfaces and a line. In this circumstance we have two possible arrangements, $(D_\theta \circ D_\sigma)(L_g)$ and $D_\sigma(D_\theta(L_g))$ as depicted in Figure 8. As D_μ is a bijective morphism $\forall \mu \in H$, these two arrangements should be equivalent. That is,

$$(D_\theta \circ D_\sigma)(L_g) = D_\sigma(D_\theta(L_g)).$$

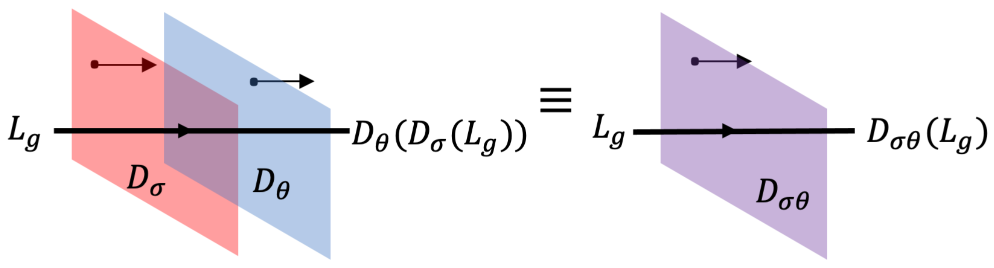

Figure 8. Two surfaces, one line

Defects as structured by 2-categories

Researchers including Bartsch et al.[1] have generalised the structure of defects described through a categorical lens. Consider the group of line defects expressed as a one-object category with invertible endomorphisms representing the line defect elements. Then one can extend this to a 2-category by introducing morphisms between the morphisms. If these morphisms between morphisms are made to be the surface defects, then they are invertible. This is illustrated in Figure 9.

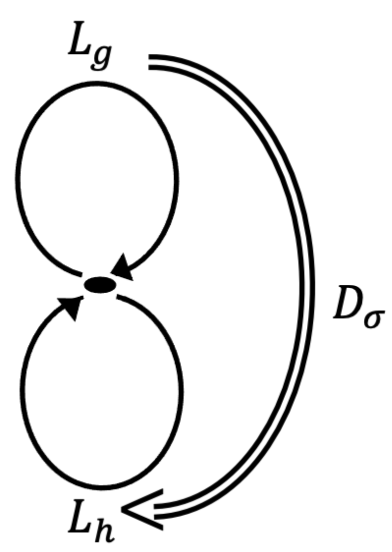

Figure 9. 2-category structure

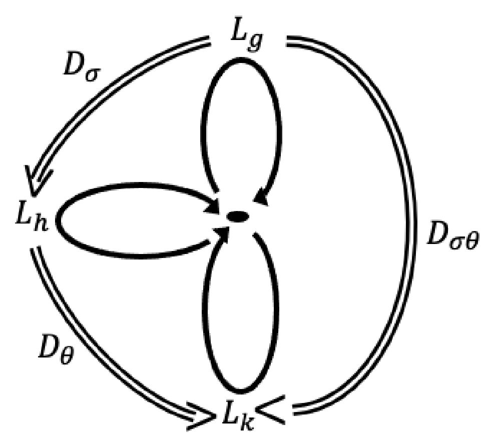

Figure 10. 2-category vertical composition

Under this construction, each of the prior consistency conditions hold, with surface defect fusion being equivalent to vertical composition of the 2-morphisms (Figure 10) and line defect fusion equivalent to composition of the 1-morphisms. There is, however, an additional composition in 2-categories, with horizontal composition of 2-morphisms (Figure 5). The physical interpretation of this is depicted in Figure 6.

This is equivalent to an operation $\square$ between surface defects that fulfills

$$(D_\sigma(L_g)) * (D_\theta(L_h)) = (D_\sigma \square D_\theta)(L_{gh})$$

which is found by equating L_k in Figure 6.

References

[1] T Bartsch, M Bullimore, AEV Ferrari, and J Pearson. Non-invertible symmetries and higher representation theory i. *arXiv:2208.05993v2 [hep-th]*, 2023.

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