Geometric properties of the n-line

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Introduction

Upon finishing high school, Ian Macdonald wrote a paper, orginally meant to be a supplementary chapter to a textbook that has been lost, concerning some results in Euclidean geometry. This paper was never finished or published. This led him to some important results in Algebraic Geometry, namely proving the Weil Conjectures for symmetric products. A few of the results in this unpublished paper are the subject of this poster.

The n-line and its results

An *n-line* is a collection of n lines in the plane, $\mathcal{L} = \{\ell_1, \dots, \ell_n\}$, with two properties:

1. No two lines are parallel
2. No three lines are concurrent

First a simple theorem:

Theorem 1 (Miquel, [Miq36])**.** *Given a three-line, $\mathcal{L} = \{l_1, l_2, l_3\}$, and three points P_1, P_2, P_3, where $P_1 \in \ell_1, P_2 \in \ell_2, P_3, \in \ell_3$, then the circles $A_{12}P_1P_2, A_{13}P_1P_3, A_{23}P_2P_3$ meet at a point.*

Inverting this result around any point yields the following equivalent results:

Theorem 2. *Given three circles c_1, c_2, c_3 with a point Q such that $c_1 \cap c_2 = \{Q, A_3\}, c_1 \cap c_3 = \{Q, A_2\}, c_2 \cap c_3 = \{Q, A_1\}$ Given $P_1 \in c_1, P_2 \in c_2, P_3 \in c_3$, the three circles $A_1P_2P_3, A_2P_1P_3, A_3P_1P_2$ meet at a point K.*

Theorem 3. *Given a circle or line c_3 and four points $A_2, Q, A_1, P_3 \in c_3$ and four circles $c_1, c_2, \Gamma_1, \Gamma_2$, such that $A_2, Q \in c_1, Q, A_1 \in c_2, A_1, P_3 \in \Gamma_1, P_3, A_2 \in \Gamma_2$ and $c_1 \cap c_2 = \{Q, A_3\}, c_2 \cap \Gamma_1 = \{A_1, P_2\}, \Gamma_1 \cap \Gamma_2 = \{P_3, K\}, \Gamma_2 \cap c_1 = \{A_2, P_1\}$ Then P_1, P_2, K, A_3 are either collinear or concyclic.*
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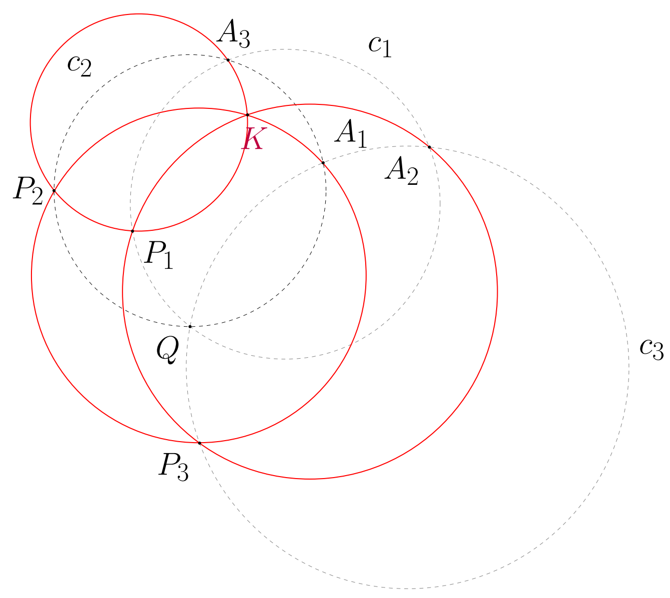

Figure 1: Theorem 2 and Theorem 3

Clifford's Chain

Two non-parallel lines intersect at a point. With a three-line, there are three such intersection points, and these intersection points lie on a circle, the circumcircle of the three-line. With a four-line, there are four such circumcircles, and these four circles meet at a point, known as the Wallace point of the Four-Line [Wal06]. With a five-line, there are five such Wallace points, and these lie on a circle, known as the Miquel circle. This chain, in most cases, continues indefinitely. More formally:

Definition. *Given an n-line, $\mathcal{L} = \{\ell_1, \dots, \ell_n\}$, where $n \geq 2$ and let $\mathcal{L}_i = \mathcal{L} \backslash \{\ell_i\}$.*

1. *if $n = 2$ then the* Clifford point *of $\mathcal{L}$ is contained in $\ell_1 \cap \ell_2$*
2. *If n is odd, then the* Clifford circle *of $\mathcal{L}$ is the circle which contains the Clifford points of $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n$*
3. *If n is even and greater than 2, then the* Clifford point *of $\mathcal{L}$ is the point contained in the intersection of the Clifford circles of $\mathcal{L}_1, \dots, \mathcal{L}_n$*

The existence of a Clifford point or circle requires a theorem.

Theorem 4 (Clifford, [Cli70])**.** *Given an n line, $\mathcal{L}$ with $n \geq 2$*

1. *if n is even then the Clifford circles of $\mathcal{L}_1, \dots, \mathcal{L}_n$ meet at a point*
2. *if n is odd then the Clifford points of $\mathcal{L}_1, \dots, \mathcal{L}_n$ lie on a circle.*

Here we shall only show that for a given $k \in \mathbb{Z}_{\geq 3}$, Clifford's chain holding for $n = 2k$ implies that Clifford's chain holds for $n = 2k + 1$. Let A be a subset of $\{1, 2, \dots, n\}$ and let (A) (or $\{A\}$) denote the Clifford circle (or point) associated with $\mathcal{L} \backslash A$. In *Figure 3*, a line between Clifford point and a Clifford circle indicates that the point lies on the circle. The inductive hypothesis gives the black lines, which is the configuration in *Theorem 3*. Hence, the Clifford points of $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$ lie on a circle, yielding the red lines. Iterating this over all subsets of size 4 of $\{1, 2 \dots, n\}$ shows Clifford's chain holds for $n = 2k + 1$. The rest of the proof is similar in nature.

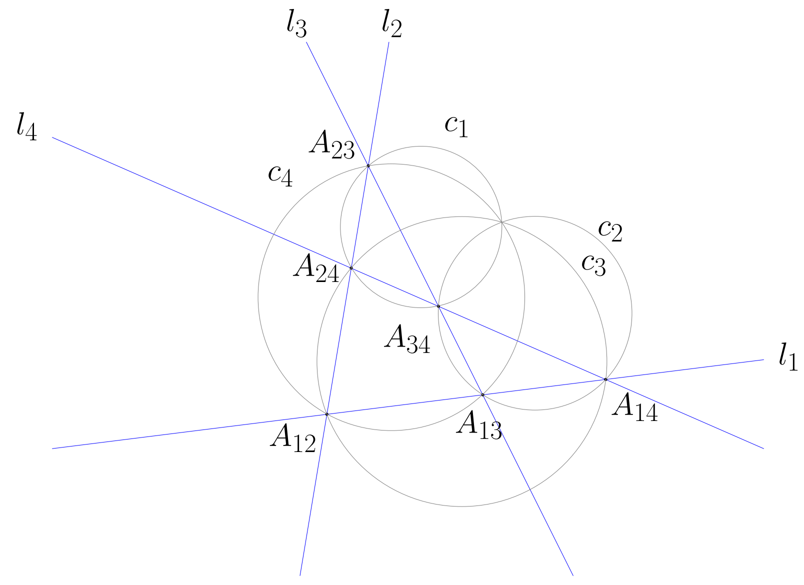

Figure 2: Wallace's Theorem, case $n = 4$ of *Theorem 4*

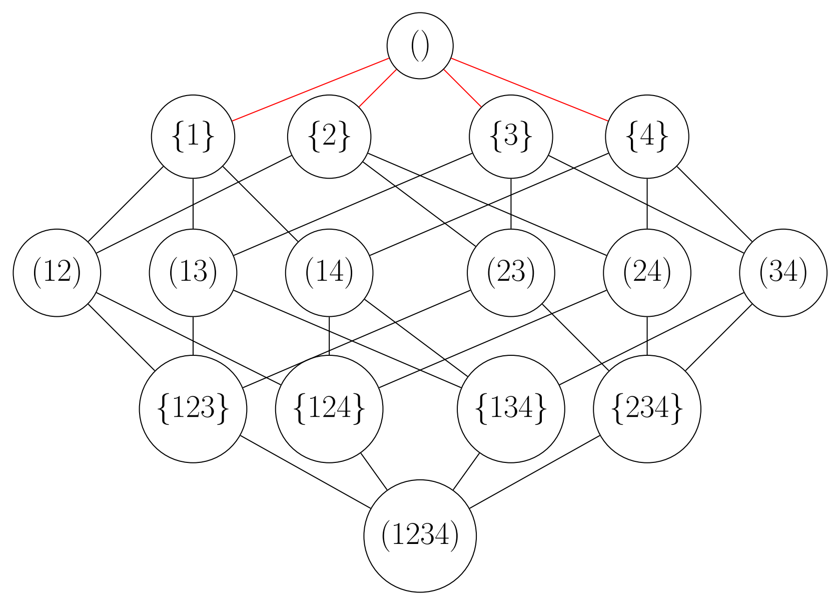

Figure 3: The inclusion of Clifford points in Clifford circles

This argument is very general and thus a similar method can be employed to obtain other results.

De Longchamps' Chain

If we consider a three-line, as before we have the circumcircle of the three-line. Given a four-line, for each subset of three lines there is a circumcircle. There are four such circumcircles and the centres of these four circumcircles, the circumcentres, lie on a circle, the Steiner circle of the four-line [Ste27]. Given a five-line, there are five such Steiner circles and the centres of these Steiner circles lie on a circle. This continues indefinitely. More formally:

Definition. *Given an $n \geq 3$ line $\mathcal{L} = \{\ell_1, \dots, \ell_n\}$, let $\mathcal{L}_i = \mathcal{L}_n \backslash \ell_i$,*

1. *If $n = 3$ the* centric circle *of $\mathcal{L}$ is the circumcircle*
2. *If $n > 3$, the* centric circle *of $\mathcal{L}$ is the circle which contains the centres of the centric circles of $\mathcal{L}_1, \dots, \mathcal{L}_n$*

Theorem 5 (de Longchamps, [DeL77])**.** *Let $\mathcal{L}$ be an n-line.*

1. *The centric circles of $\mathcal{L}_1, \dots, \mathcal{L}_n$, meet at a point*
2. *Let C_i be the centre of the centric circle of $\mathcal{L}_i$. Then, there exists a circle K such that $C_1, \dots, C_k \in K$*

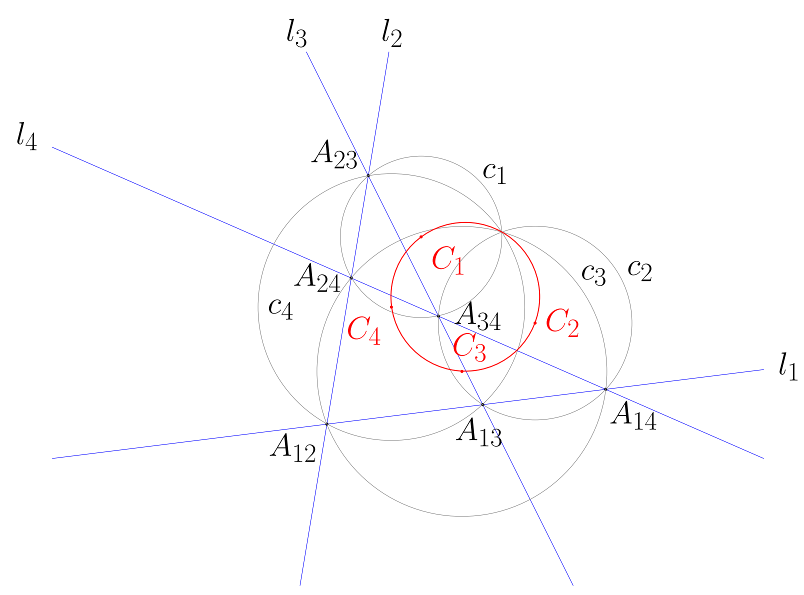

Figure 4: Steiner's Theorem, case $n = 4$ of *Theorem 5*

An Analytic Approach to the n-line

Alternatively we can think of lines as sets of complex numbers. Any line that does not pass through the origin can be written in the form [Mor00]

$$\ell_i = \{x \in \mathbb{C} \mid \bar{x} = t_i(x - p_i)\}$$

Where $p_i \in \mathbb{C}^\times$ and $t_i = \frac{-\bar{p}_i}{p_i}$. This line is the set of points equidistant to 0 and p_i. Hence, an n-line $\mathcal{L}$ is described algebraically. In this set up we can again prove Clifford's Chain and De Longchamp's Chain. Define for an n-line $\mathcal{L}$ and $i, r \in \{1, \dots, n\}$

$$f_i(\mathcal{L}) = \prod_{j \in \{1, \dots, i-1, i+1, \dots, n\}} (t_i - t_j)$$

$$s_r(\mathcal{L}) = \sum_{1 \leq j_1 < \dots < j_r \leq n} t_{j_1} t_{j_2} \dots t_{j_k}$$

And subsequently for $k \in \{0, 1, \dots, n-1\}$

$$c_k(\mathcal{L}) = \sum_{i=1}^{n} \frac{x_i t_i^{n-1-k}}{f_i(\mathcal{L})}$$

Hence,

Theorem 6. *Given an n-line $\mathcal{L}$ and $0 \in \{1, \dots, n-1\}$*

$$\bar{c}_k(\mathcal{L}) = (-1)^n s_n(\mathcal{L}) c_{n-1-k}(\mathcal{L})$$

Theorem 7. *Given an n-line $\mathcal{L}$ and $k \in \{0, \dots, n-2\}$. For $j \in \{1, \dots, n\}$, let $\mathcal{L}_j = \mathcal{L} \backslash \{\ell_j\}$, then*

$$c_k(\mathcal{L}_j) = c_k(\mathcal{L}) - c_{k+1}(\mathcal{L}) t_j$$

From these identities De Longchamps' and Clifford's Chain follow.

Theorem 8. *The centric circle of $\mathcal{L}$ is*

$$K = \{x \in \mathbb{C} | x = c_0(\mathcal{L}) - c_1(\mathcal{L})\tau|, \tau \in \mathbb{C}, |\tau| = 1\}$$

Theorem 9. *Let $\mathcal{L} = \{\ell_1, ..., \ell_n\}$, where $n = 2k$ is even. Then the Clifford point of $\mathcal{L}$ is given by*

$$x = \frac{\begin{vmatrix} c_0(\mathcal{L}) & \dots & c_{k-1}(\mathcal{L}) \\ \vdots & \ddots & \vdots \\ c_{k-1}(\mathcal{L}) & \dots & c_{2k-2}(\mathcal{L}) \end{vmatrix}}{\begin{vmatrix} c_2(\mathcal{L}) & \dots & c_k(\mathcal{L}) \\ \vdots & \ddots & \vdots \\ c_k(\mathcal{L}) & \dots & c_{2k-2}(\mathcal{L}) \end{vmatrix}}$$

If $n = 2k + 1$ is odd, the Clifford circle of n is given by

$$\left\{ \frac{\begin{vmatrix} c_0(\mathcal{L}) & \dots & c_{k-1}(\mathcal{L}) \\ \vdots & \ddots & \vdots \\ c_{k-1}(\mathcal{L}) & \dots & c_{2k-2}(\mathcal{L}) \end{vmatrix}}{\begin{vmatrix} c_2(\mathcal{L}) & \dots & c_k(\mathcal{L}) \\ \vdots & \ddots & \vdots \\ c_k(\mathcal{L}) & \dots & c_{2k-2}(\mathcal{L}) \end{vmatrix}} - \frac{\begin{vmatrix} c_1(\mathcal{L}) & \dots & c_k(\mathcal{L}) \\ \vdots & \ddots & \vdots \\ c_k(\mathcal{L}) & \dots & c_{2k-1}(\mathcal{L}) \end{vmatrix}}{\begin{vmatrix} c_2(\mathcal{L}) & \dots & c_k(\mathcal{L}) \\ \vdots & \ddots & \vdots \\ c_k(\mathcal{L}) & \dots & c_{2k-2}(\mathcal{L}) \end{vmatrix}} \tau : |\tau| = 1 \right\}$$

In fact we need the determinants in the denominators to be non-zero. This is a condition which is not always met. A geometric equivalent to this can be found in [Car20].

Acknowledgements

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