Moduli space of spatial polygons

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2023/2024 Mathematics and Statistics Vacation Scholarships Program, The University of Melbourne

Introduction

This project will explore the volume form of a n-sided polygon in $\mathbb{R}^3$ under $SO(3)$ group. We will start by formulating the problem into a well-known solution, and then use symplectic reduction to achieve our goal. Ultimately, we will give an explicit symplectic (volume) form of the space.

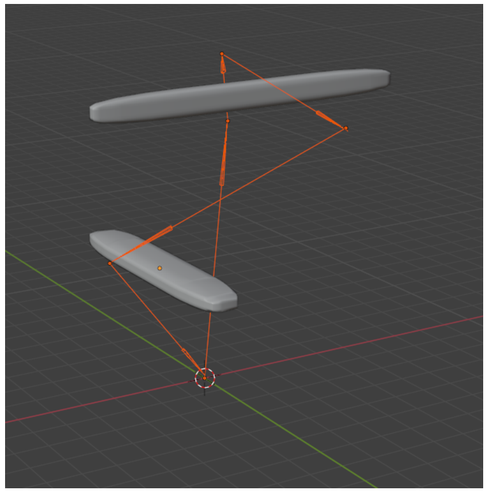

Figure 1:Example of 5-sided polygon in $\mathbb{R}^3$

Background

- Symplectic geometry
  A symplectic manifold is a pair (M, ω) where M is a smooth manifold and ω is a closed and non-degenerate 2-form on M.
- Symplectic reduction
  Let (M, ω) be a symplectic manifold with a Hamiltonian action of compact Lie group G and moment map $\mu : X \to \mathfrak{g}^* = Lie(G)^*$. If 0 is a regular value of μ s.t. G acts freely on $M = \mu^{-1}(0)$, then M/G is a symplectic manifold with symplectic structure induced by ω.
- Moment map
  Let (M, ω) be symplectic manifold and a Lie group G acts on M via symplectomorphisms. Then any $\xi \in \mathfrak{g}$ induces a vector field to describe the infinitesimal action of ξ. The vector at $x \in M$ is:

$$\frac{d}{dt}\Big|_{t=0} exp(t\xi) \cdot x$$

The moment map for G-action is $\mu : M \to \mathfrak{g}^*$ s.t.

$$d(\langle \mu, \xi \rangle)_e = \omega(\tfrac{d}{dt}\Big|_{t=0} exp(t\xi) \cdot e, \cdot)$$

Moduli space of spatial polygons

Take $r = (r_1, ..., r_n)$ as a tuple of positive numbers and consider an n-sided polygon in $\mathbb{R}^3$ with each edge v_i has length r_i. The space of all such polygons is denoted as $M(r)$. It's easy to see that the edge v_i correspond to a vector in 2-sphere S_{r_i}, which gives a map:

$$\mu : S_{r_1} \times ... \times S_{r_n} \to \mathbb{R}^3$$
$$(v_1, ..., v_i) \mapsto v_1 + ... + v_i.$$

Notice that 0 is a regular value of μ if there is no solution to $\epsilon_1 r_1 + ... + \epsilon_n r_n = 0$ with $\epsilon \in \{\pm 1\}$, and we will assume this in the following text.
A polygon satisfies $v_1 + ... + v_n = 0$. So $M(r) = \mu^{-1}(0)$ which is a submanifold of $S_{r_1} \times ... \times S_{r_n}$ of dimension $2n-3$ (S_{r_i} has dimension 2).
Besides, we want the polygons to be invariant under rotation. So the space we are interested in is $\mathcal{M}(r) := \mu^{-1}(0)/SO(3)$, which is a manifold of dimension $2n-6$.

Moment mapping for $SO(3)$

Consider the sphere S_r of radius r in $\mathbb{R}^3$. We give it the symplectic form that is $1/r$ times the usual area form:

$$\omega_e(v, w) = \frac{det(v, w, e)}{r^2}$$

with $e \in S_r$, $v, w \in T_e(S_r)$. Since $SO(3)$ acts on symplectic manifold via sympletomorphisms, this gives a collection of flows via $exp(t\xi)$ for $\xi \in \mathfrak{so}(3)$.
One can show that $\mu : S_r \to \mathfrak{so}(3)^* \backsimeq (\mathbb{R}^3)^* \backsimeq \mathbb{R}^3$ is a moment mapping for $SO(3)$ action on S_r, by showing the following equality holds:

$$(d\langle \mu, \xi \rangle)(v) = \omega_e(\frac{d}{dt}\Big|_{t=0} exp(t\xi)e, v)$$

where $\langle \mu, \xi \rangle$ is a Hamiltonian function and $\langle ., . \rangle$ is the standard inner product in $\mathbb{R}^3$, $v \in T_e S_r$.
As a result, we have:

$$\langle v, \xi \rangle = \frac{det(\xi \times e, v, e)}{||e||^2}$$

and the moment map for $SO(3)$ action on $S_{r_1} \times ... \times S_{r_n}$ of spheres is the addition map μ.

The symplectic structure on $\mathcal{M}(r)$

The symplectic space $(S_{r_1} \times .. \times S_{r_1}, \omega)$ with a moment map μ guarantee a symplectic structure on $\mathcal{M}(r)$ by symplectic reduction. To compute the symplectic pairing of two tangent vectors to $\mathcal{M}(r)$ we lift them to $S_{r_1} \times ... \times S_{r_n}$ and take the pairing there.
Consider an oriented polygon in $\mathbb{R}^3$ given by $(v_1, ..., v_n) \in \Pi S_{r_i}$, define the length $\ell_i = ||v_1 + ... + v_{i+1}||$. Let $\mathcal{M}'(r) \subset \mathcal{M}(r)$ be the locus where all ℓ_i are non-zero and each $v_1 + ... + v_i$ is not co-linear to v_i. Then we can define triangle T_i that span the initial vertex and the $v_1 + ... + v_i$ and $v_1 + ... + v_{i+1}$ edge. Define θ_i to be the dihedral angle from T_i to T_{i+1}. Then the $\{\ell_i, \theta_i\}_{i=1,..,n-3}$ are smooth coordinates for $\mathcal{M}'(r)$.

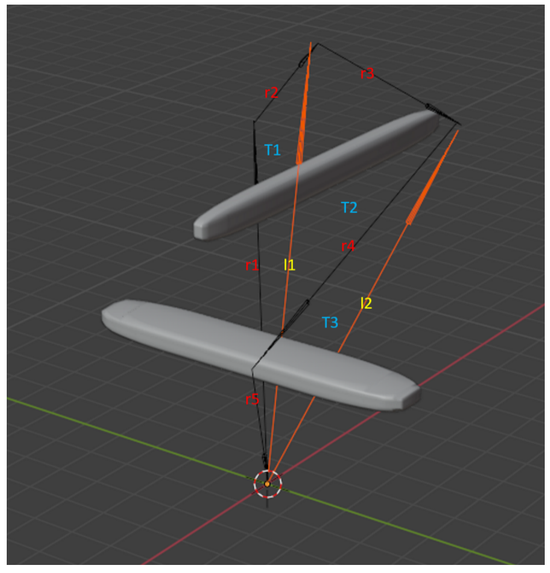

Figure 2:Example of 5-sided polygon in $\mathbb{R}^3$

For each i, there exists a bending flow that twists the first part of the polygon around the i-th diagonal. This fixes every coordinate except θ_i. Lifting to $\mu^{-1}(0)$ the flow rotates $v_1, ..., v_{i+1}$ about the direction of $v_1 + ... + v_{i+1}$, which one can prove is the Hamiltonian flow for the function ℓ_i. Besides, this preserves $\mathcal{M}'(r)$ except θ_i, on which it acts by $\theta_i \mapsto \theta_i + t$, or:

$$d\ell_i(\cdot) = \omega(\frac{\partial}{\partial \theta_i}, \cdot)$$

Volume form of $\mathcal{M}(r)$

Using the results one can show that the symplectic form on $\mathcal{M}'(r)$ is:

$$\omega = d\theta_1 d\ell_1 + ... + d\theta_n d\ell_n$$

and as a result the volume of $\mathcal{M}(r)$ is $(2\pi)^{n-3}$ times the usual Euclidean volume of the image the map $(\ell_1, ..., \ell_{n-3})$.
One can see that the volume is a piecewise polynomial. There exists a beautiful closed form for the volume of $\mathcal{M}(r)$[1]:

$$vol\mathcal{M}(r) = -\frac{(2\pi)^{n-3}}{2(n-3)!} \sum_{\text{I long}} (-1)^{n-|I|} \epsilon_I(r)^{(n-3)}$$

with $I \subseteq \{1, 2, ..., n\}$ and

$$\epsilon_I = \sum_{i \in I} r_i - \sum_{i \in \{1,..,n\} \setminus I} r_i$$

, I is long if $\epsilon_I > 0$.

Further extensions

- Establish the relationship between the two symplectic forms.
- Show that the volume form is independent of the order of the edge
- Show that the volume is independent of the choice of the ℓs
- Explore the factorization property of the volume form under different chamber conditions

References

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