Riemann surfaces and the Hurwitz existence problem

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Riemann surfaces and finite maps

A **Riemann surface** is a Hausdorff, connected topological space, together with an atlas of charts to the complex plane with holomorphic transition functions.

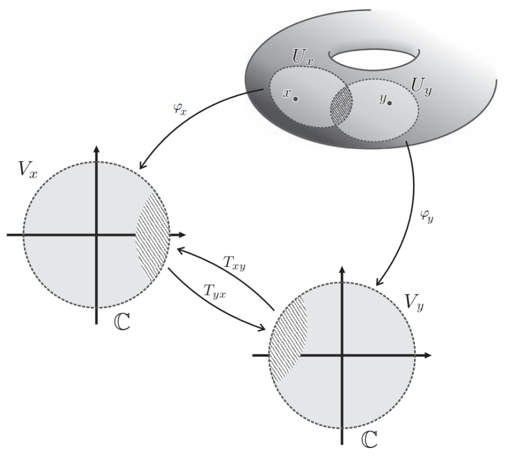

Figure 1. A diagram of a Riemann surface showing the transition function of two charts in the complex plane. From [2]

A **finite map** $f : X \to Y$ between compact Riemann surfaces is a nonconstant holomorphic map such that every $y \in Y$ has a finite preimage $f^{-1}(y)$. The size of each preimage is fixed for all but a finite set of points, called the **branch points**, as in Figure 2.
f can be extended through local coordinates to be a complex function f^*. When there exist local coordinates around a point $x \in X$ such that $f^*(z) = z^{r_x}$ for **local ramification** $r_x > 1$, x is called a **ramification point**. All ramification points are contained in the preimage of some branch point.

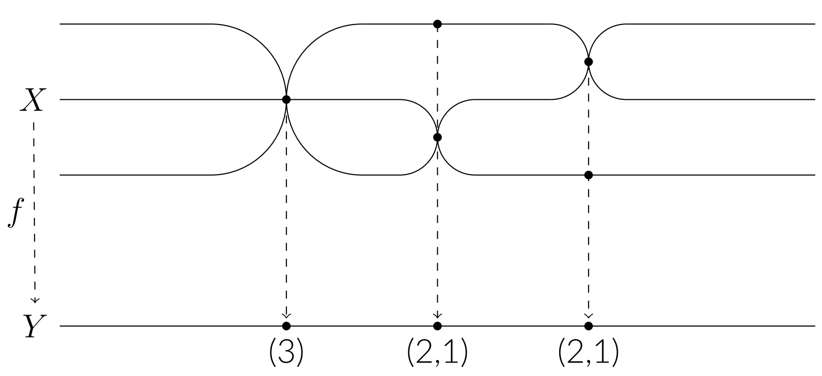

Figure 2. A diagram of a degree 3 finite map, with labels of branch points in Y describing the ramification above them: $\mathcal{D} = \{(3), (2,1), (2,1)\}$

The set of partitions describing ramification for a finite map is called a branch datum.

The Riemann-Hurwitz formula

Finite maps between Riemann surfaces have some constraints applied to them, the first and most important being the **Riemann-Hurwitz formula**. A finite map $f : X \to Y$ between Riemann surfaces of degree d must have

$$2g(X) - 2 = d(2g(Y) - 2) + \sum_{x \in X}(r_x - 1) \qquad (1)$$

The Riemann-Hurwitz formula can be rewritten in terms of the branch datum $\mathcal{D}$ associated with the finite map.
The Riemann-Hurwitz formula applies some key restrictions on the possible sets of partitions describing a finite map, called the Hurwitz conditions. They are

- $R(\mathcal{D}) = \sum_{x \in X}(r_x - 1)$ is even, ensuring $g(X)$ is an integer
- $R(\mathcal{D}) \geq 2d(1 - g(Y)) - 2$, ensuring $g(X)$ is non-negative

A branch datum $\mathcal{D}$ is **compatible** when it satisfies the Hurwitz conditions.

Monodromy representations of finite maps

A loop in Y around a branch point and based at $y_0 \in Y$ can be "lifted" up through the finite map to a *path* in X, as in Figure 3. This can be seen to join or permute elements of $f^{-1}(y_0)$.

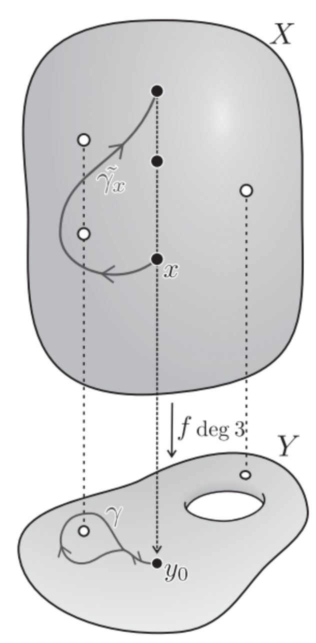

Figure 3. An example of a loop in Y based at y_0 being lifted through a finite map to X. From [2]

The monodromy representation of the finite map is the group homomorphism

$$\Phi : \pi_1(Y \backslash B, y) \to S_d \qquad (2)$$

that sends loops around branch points to the permutations obtained by this lift.

The Hurwitz existence problem (1891)

The Hurwitz existence problem asks for which sets of partitions $\mathcal{D}$ describing potential ramification does there exist a finite map $f : X \to Y$ **realising** this ramification profile.
The problem was first posed by Hurwitz in [4] and has attracted significant interest due to results such as Belyi's theorem on algebraic curves.

Non-spherical target surfaces

When the target surface Y has a genus $g(Y) > 1$ the Hurwitz conditions are sufficient for realisability [3].
The remaining cases for study are when the target surface is a sphere. In these cases, the Hurwitz conditions reduce further to $R(\mathcal{D})$ being even and $R(\mathcal{D}) \geq 2d - 2$.

Previous strategies

- Study the properties of the monodromy representation associated with the finite map. This technique was used in [3], leading to key results and the **prime degree conjecture**: that the Hurwitz conditions are sufficient when the degree d of the map is prime.
- Analyse the types of **dessins d'enfants** ("child's drawing") that can be embedded on the covering surface X, first done in [5]. This technique is used to relate the realisability of similar but distinct branch data.

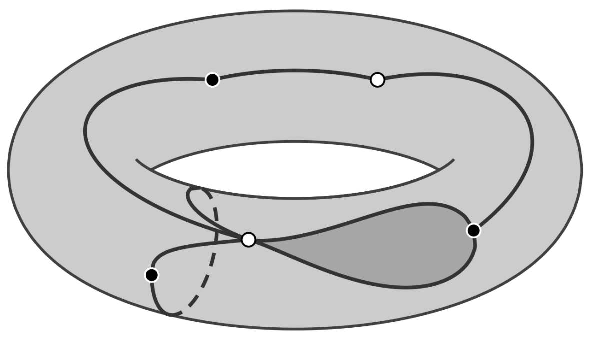

Figure 4. A dessins d'enfants upon a torus. From [1]

- A combination of these two techniques was used in [1] to fully classify branch data containing a partition of length 2.
- Other research has transferred the general structure of Riemann surfaces into other, more restricted structures such as orbifolds and cone metrics to obtain further partial results.

Some new results

Theorem: Consider a general compatible branch datum $\mathcal{D}$ containing n partitions of degree d. Either its realisability can be directly confirmed, or there exists a related branch datum $\mathcal{D}'$ containing 3 partitions of degree d such that the realisability of $\mathcal{D}'$ is sufficient for $\mathcal{D}$ to be realisable.

This theorem is a direct result of combining several of the techniques developed in [1]. Importantly, it reduces in part the general set of unknown branch data to the set of data containing three partitions. A similar result was shown to be true of the prime degree conjecture in [3].
A subset of branch data with three partitions, one of which being of the form $(a, b, 1, \ldots, 1)$, has been classified for realisability. This builds on previous research ([3]) that fully classified the branch data with one partition of the form $(m, 1, \ldots, 1)$.

Flat diagrams of the Riemann sphere

Flat geometry can be used to create a new way of representing a finite cover of a sphere. A Riemann sphere can be represented as an infinite cylinder with points at infinity in each direction. We can then consider degree d covers of the sphere branched over those two points as d copies of the infinite cylinder, joined together by horizontal edges differently on each side of the preimages of the third branch point, represented by the split in Figure 5. Cone points of angle $2\pi m$ give ramification points of order $m - 1$.

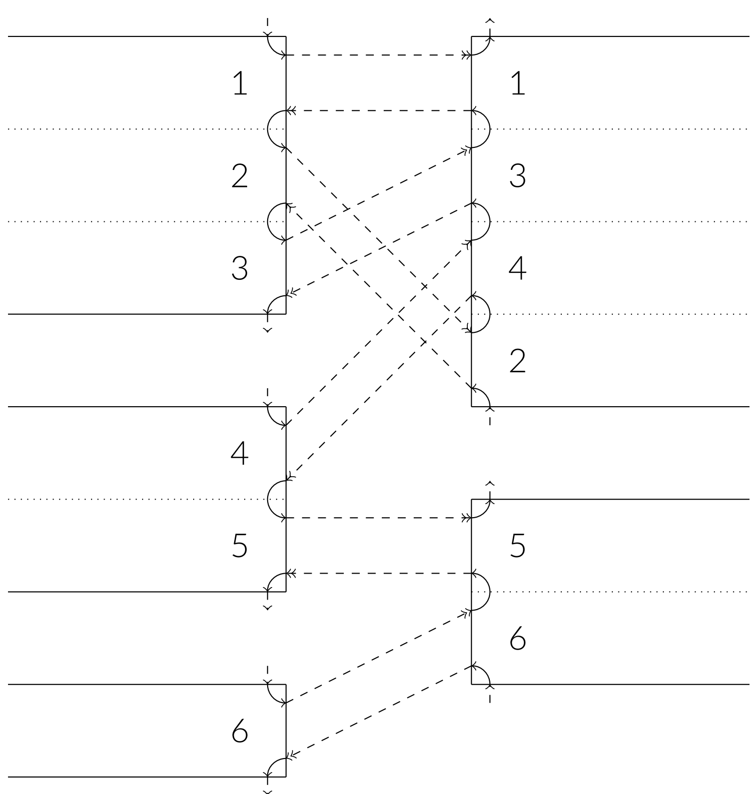

Figure 5. A flat diagram of a degree 6 cover of a sphere by a torus, with ramification $\mathcal{D} = \{(3,2,1), (4,2), (6)\}$.

Generating realisable data

Given a realisable branch datum with three partitions, any odd number can be appended to one and added to members of the other two by the process shown in Figure 6. For instance, consider the realisable branch datum $\mathcal{D} = \{(3), (2,1), (2,1)\}$ and insert three rows to produce realisable $\mathcal{D}' = \{(3, \mathbf{3}), (2{+}\mathbf{3}, 1), (1{+}\mathbf{3}, 2)\} = \{(3,3), (5,1), (4,2)\}$.

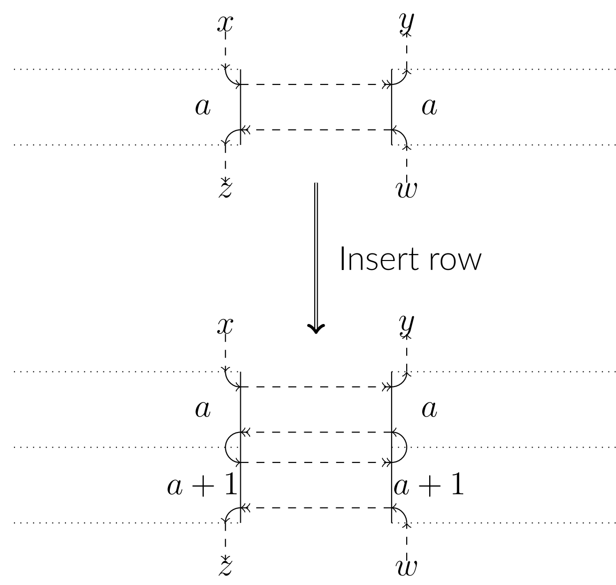

Figure 6. The insertion of one new rectangle in a flat diagram. Any odd number of new rectangles can be inserted. This action corresponds with appending as above.

Generating unrealisable data

Flat diagrams also identify certain **factorings** of finite maps by mapping to intermediate spheres, as in Figure 7. If there is a way to factor a potential finite map f with branch datum $\mathcal{D}$ into $g \circ h$ such that the branch data of g or h is unrealisable, then $\mathcal{D}$ is also unrealisable.

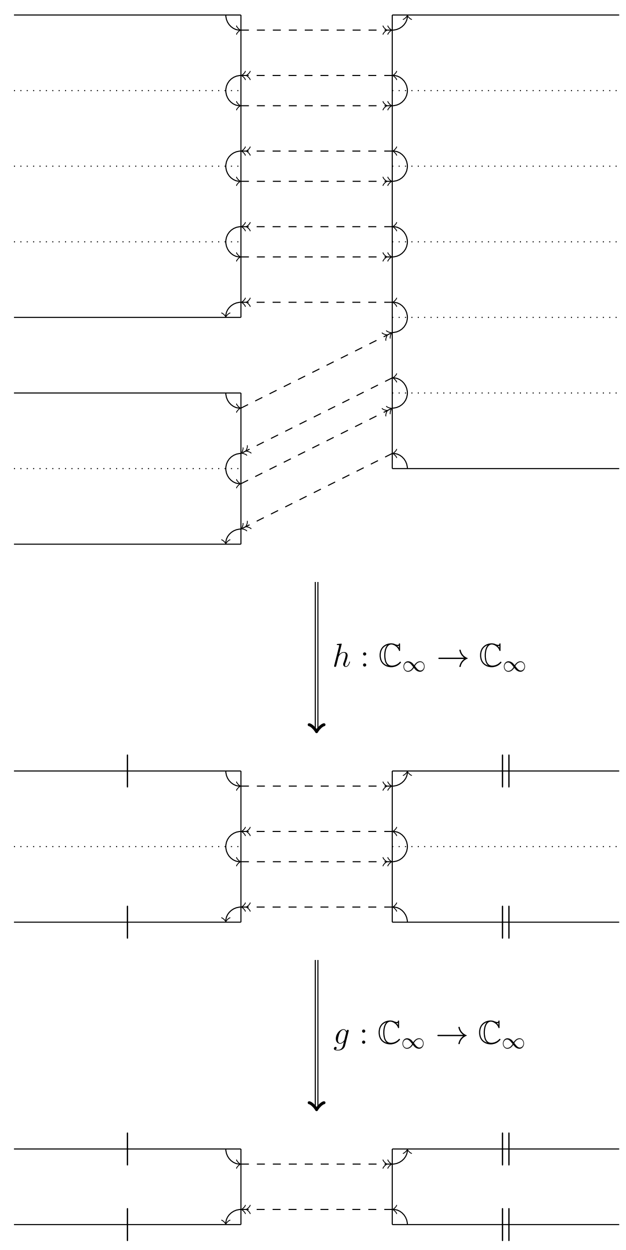

Figure 7. A diagram of a finite map $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ with $\mathcal{D} = \{(6), (4,2), (2,1,1,1,1)\}$ shown to be factored as $f = g \circ h$.

By reversing this process, each unrealisable branch datum representing a potential map f can generate an infinite family of unrealisable branch data that would contain a factor of f. Checking of all unrealisable data up to $d = 10$ showed 51 out of the 59 branch data could be generated this way. Further, this result produces unrealisable data only of composite degree, in agreement with the prime degree conjecture.

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