Permutations avoiding patterns under iterates of the fundamental bijection

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Introduction

We are interested in enumerating permutations which avoid certain *patterns*: strict total orders on a subset of the entries.

- The *one-line* notation of a permutation $\pi \in \mathcal{S}_n$ is $\pi = \pi_1\pi_2\ldots\pi_n$, where $\pi_i = \pi(i)$ for $1 \leq i \leq n$.
- We use *standard cycle notation* to mean the expression of a permutation as a product of disjoint cycles, written with their largest elements first and in increasing order.

A pattern of length k is denoted by a corresponding element of $\mathcal{S}_k$, e.g. the pattern 132 is found in the (one-line) permutation $5\underline{14}3\underline{2}$.

The fundamental bijection

The *fundamental bijection* $\theta : \mathcal{S}_n \to \mathcal{S}_n$ is constructed as follows: given a permutation π in standard cycle notation, remove the parentheses; the result can be interpreted as another permutation in one-line notation, which we identify as $\theta(\pi)$.

For example, $326451 = (2)(4)(5)(6\ 1\ 3)$, so $\theta(326451) = 245613$.

One question that can be asked is: for a given pattern σ, which permutations $\pi \in \mathcal{S}_n$ satisfy that both π and $\theta(\pi)$ avoid σ?

A natural extension of this is to iterate θ further, and ask (for some positive integer $k \geq 2$) which permutations $\pi \in \mathcal{S}_n$ satisfy that $\theta^m(\pi)$ avoids σ for all integers $0 \leq m \leq k$. We denote this set by $\mathcal{T}_n^k(\sigma)$, and its cardinality by $t_n^k(\sigma)$.

For example, 54321 and $\theta(54321) = 34251$ avoid 132, but $\theta^2(54321) = 5\underline{132}4$ does not, so $54321 \in \mathcal{T}_5^1(132)$ but not $\mathcal{T}_5^2(132)$.

Specialising to $\sigma \in \mathcal{S}_3$

When $\sigma \in \mathcal{S}_3$, the above two questions are mostly answered [1]; we summarise the results in the following table.

σ	1	2	3	4	≥ 5
213	$2F_{n+2} + n^2 - 6n + 4$	$\binom{n+1}{2}$	$2n+1$	$n+4$	7
231	2^{n-1}	F_{n+1} for all $k \geq 2$			
312	2^{n-1}	F_{n+1} for all $k \geq 2$			
321	$[x^n]\dfrac{2x^2}{2x(1-x) - 1 + \sqrt{1-4x^2}}$	F_{n+1} for all $k \geq 2$			
123	0 for all k				

Table 1. Values of $t_n^k(\sigma)$ for $n \geq 11$. (For $\sigma = 321$, only the o.g.f. is known.)

The remaining case

It remains to settle $\sigma = 132$, the main result of this project. In the enumeration of $\mathcal{T}_n^1(132)$ in [1], several useful facts are discovered:

(1) If $\pi \in \mathcal{T}_n^1(132)$, then either π is cyclic, $\pi(n) = n$, or $\pi(n) = 1$, $\pi(1) = n$.
(2) If $\pi \in \mathcal{T}_n^1(132)$ is cyclic, either $\pi = (n\ 1\ 2\ \ldots\ n-1)$ (a special permutation we call α_n), or $\pi(1) = n$
(3) If $\pi \in \mathcal{T}_n^1(132)$ is cyclic and $\pi(1) = n$, either $\pi(n) = n-1$ or $\pi(n-1) = 1$.
(4) In the first case of (3), deleting the n (in standard cycle notation) yields an element of $\mathcal{T}_{n-1}^1(132)$, and all such elements (**except** α_{n-1}) are attainable in this fashion.
(5) In the second case of (3), deleting the $n-1$ and 1 (in standard cycle notation) and shifting values down to the set $\{1, 2, \ldots, n-2\}$ yields an element of $\mathcal{T}_{n-2}^1(132)$, and all such elements are attainable in this fashion.

Preparing for a recurrence

Let $f(n)$ be the number of cyclic permutations in $\mathcal{T}_n^2(132)$. Because of (1), since it can be shown that $\theta^{-1}(\alpha)$ is the only permutation containing $(n\ 1)$ in $\mathcal{T}_n^2(132)$, we find that $t_n^2(132) = t_{n-1}^2(132) + f(n) + 1$, so it suffices to find f.

It is much simpler to enumerate permutations counted by f of the form $(n\ \ldots\ n-1\ 1)$, because $\theta(\pi)$ cycle-decomposes into a sub-permutation on $\{2, 3, \ldots, n-2\}$, a 2-cycle and a 1-cycle. Applying (5), the paradigm (see next section) ensures that if the result is $(n-2\ n-3\ \ldots\ 1)$ then the original permutation was valid, while if the result is α_{n-2} then the original is in fact $\theta^{-2}(\alpha_n)$, which works too.

If it is $(n-2\ \ldots\ n-3\ 1)$, then $\pi = (n\ \ldots\ n-2\ 2\ n-1\ 1)$, so $\theta(\pi)$ has a cycle $(n-2\ 2\ \ldots\ n-3)$; for $\theta^2(\pi)$ to be 132-avoiding, this must contain all remaining elements in increasing order (which works—directly verify). Hence there are $f(n-3)+1$ such permutations.

A useful paradigm

The key ingredient in resolving the $k = 2$ case is a model for the global structure of cyclic permutations in $\mathcal{T}_n^1(132)$ as per (1)–(5):

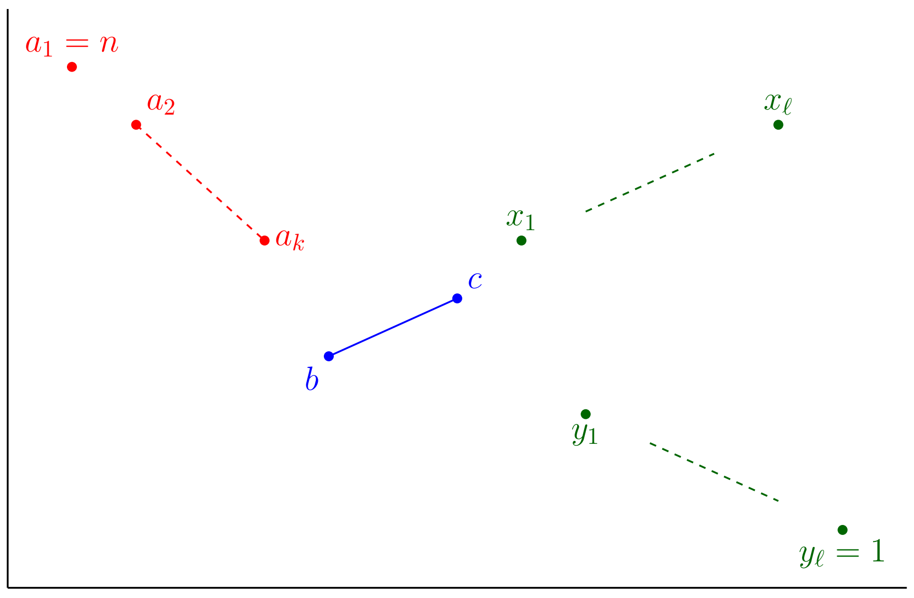


Figure 1. A visualisation of $\theta(\pi)$ for cyclic $\pi \in \mathcal{T}_n^1(132)$.

1. A decreasing sequence $(a_i)_{i=1}^k$ with $a_1 = n$
2. A non-empty sequence $b, b+1, b+2, \ldots, c < a_k$
3. Alternation between an increasing sequence $(x_i)_{i=1}^\ell$ and a decreasing sequence $(y_i)_{i=1}^\ell$, with $x_1 > c$ and $y_1 < b$ if $\ell \geq 1$.

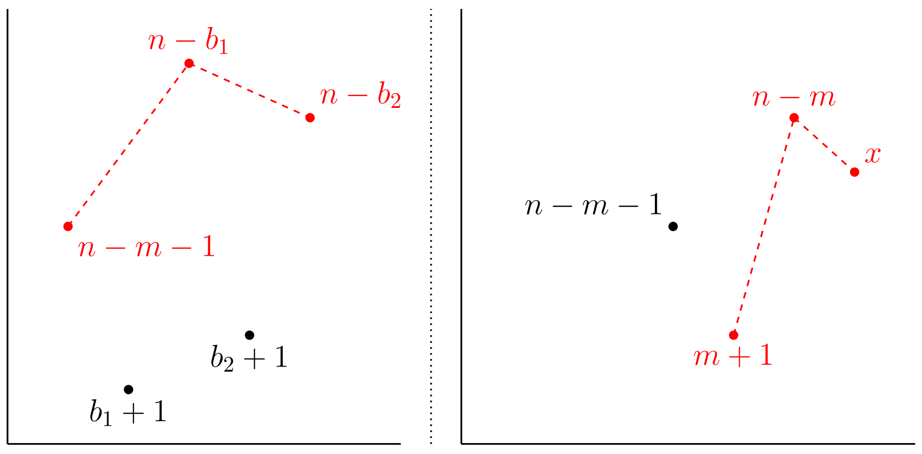
Note: If $b \neq c$ then $x_1 = c+1$ is nececessary (in particular, $\ell \geq 1$).

The key quantity m

The structure is much more difficult to elucidate for π of the form $(n\ n-1\ \ldots\ 1)$. The key quantity to categorise these by is $m(\pi)$, the largest integer such that for all $0 \leq i \leq m(\pi)$, $\theta(\pi)(i+1) = n-i$.

If $m = 1$, the paradigm ensures that if the result after (5) is not α_{n-2}, then $\theta(\pi)$ contains the cycle $(n-1\ n-2\ 2)$, which fails if $n \geq 6$. Therefore there is only one such permutation when $m = 1$.

When $m \geq 3$, we can exert a lot of control over $\theta(\pi)$. This is because we know that for $1 \leq b \leq m$, $b+1 \mapsto n-b$, and $n-1 \mapsto n-m-1$. Any element $2 \leq a < n-m-1$ is thus forced to follow $n-m-1$ in the cycle $(n-1\ n-m-1\ \ldots).$If $n > 2m+2$, this leads to a complete determination of $\theta(\pi)$, as we demonstrate pictorially:



The b's must be decreasing

All other x must lie left of $m+1$

Figure 2. Visual proofs of two properties of $\theta(\pi)$ (in cycle notation)

It follows that we also have $n - b \mapsto b$ for $2 \leq b \leq m$. Then:

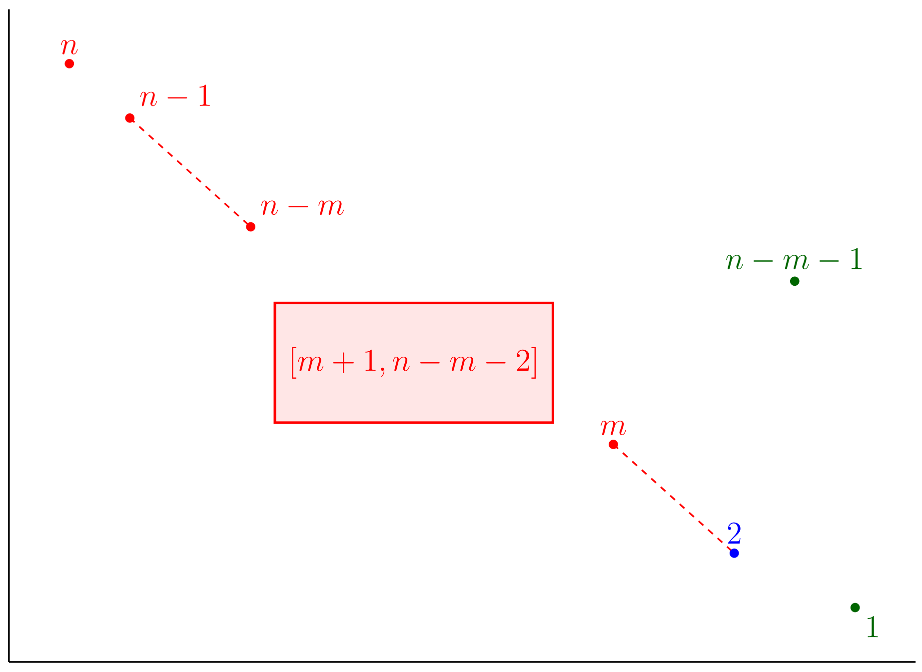


Figure 3. Visual proof (via paradigm) that $\theta(\pi)$ has at most 1 valid structure

This permutation is not cyclic if $m+1 \neq n-m-2$, because $n-m-2$ can be shown to not appear in the cycle starting with $n-1$, but works if $n = 2m+3$.

We find a partner permutation when $n = 2m+2$; on the other hand, if $n < 2m+2$, then $b+1 \mapsto n-b$ tells us too much, and a 132-pattern appears unless $n \leq m+4$, which leads to $(n\ n-1\ \ldots\ 2\ 1)$.

$m = 2$: paradigm reimagined

Assume $n \geq 8$: then all the previous discussion covers precisely the cases where $m \neq 2$, and we are charged with enumerating those permutations counted by f for which $m = 2$.

Since $m = 2$, $\pi = (n\ n-1\ n-2\ \ldots\ n-3\ 1)$, so $\theta(\pi)$ ends in $(n-1\ n-3\ \ldots\ 2)(n\ 1)$. To avoid a 132, 3 must appear in the ellipsis, and since $\theta(\pi)(3) = n-2$, to avoid a 132 in fact everything else must appear in the ellipsis too, left of $3 \mapsto n-2$, which hence points to 2.

Upon discarding $(n\ 1)$, we are left with a cyclic permutation of $\{2, 3, \ldots, n-1\}$ which belongs to $\mathcal{T}_{n-2}^1(132)$, so we seek (paradigm-following) permutations $(n\ n-1\ n-2\ \ldots\ n-3\ 1)$ such that upon removing the parentheses and the 1 and n, we have (in one-line) a cyclic permutation on $\{2, 3, \ldots, n-1\}$ whose cycle form follows the paradigm too.

The key technique here is to unwrap the paradigm in Figure 1 into the one-line notation for π by reading $\pi(1)$, $\pi(2)$, $\ldots$, $\pi(n)$.

The final stretch

The unwrapped paradigm tells us that $x_\ell = n-2$ and $a_2 = n-3$ (hence at least two a's and at least one (x_i, y_i) pair).

More importantly, if $b \neq c$ in the permutation on $\{2, 3, \ldots, n-1\}$ induced by $\theta(\pi)$, then $x_1 = c+1$ means that after x_1 (which comes after c, which is seen as $b < c$) we see y_1: this is alternating, so constitutes (x'_1, y'_1); but our next options are y_2 and b, neither of which can be x'_2, contradiction: so $b = c$.

Furthermore, once we have seen both y's and a's, they must alternate (to simulate (x'_i, y'_i) pairs). This leads to three cases:

- Some y_i's, then b, then ya pairs
- b, then some a's, then ya pairs
- b, then some y's, then ya pairs

It is possible to uniquely infer a valid π from a structure as per above, with the exception that since $a_2 = n-3$ is known, Case 3 must have at least two ya pairs.

Tallying up

The cases above give $2k-4$, $2k-4$ and $2k-3$ permutations counted by f with $m = 2$ for $n = 3k$, $n = 3k+1$, $n = 3k+2$ respectively (with $k \geq 2$ as $n \geq 8$).

Combining with the values of f for small $n \leq 7$ leads to:

Theorem 1. For $n \geq 2$, we have

$$t_n^2(132) = \begin{cases} k^3 + 3k^2 + 2k - 1 & n = 3k \\ k^3 + 4k^2 + 4k & n = 3k+1 \\ k^3 + 5k^2 + 7k + 2 & n = 3k+2 \end{cases}$$

Resolving $k \geq 3$

The case $k \geq 3$ is in fact much simpler, and can be done without knowledge of Theorem 1.

Theorem 2. For $n \geq 3$, we have the following values of $t_n^k(132)$:

k	3	4	5	≥ 6
$t_n^k(132)$	$3n-4$	$2n-1$	$n+2$	5

Table 2. Values of $t_n^k(132)$ for $n \geq 3$.

Proof. We induct on n; for $n = 3$ the answer is 5 for all $k \geq 3$ because $132 \in \mathcal{S}_3$ is a fixed point of θ. Following (1), observe that the number of $\pi \in \mathcal{T}_n^k(132)$ which have $\pi(n) = n$ will just be $t_{n-1}^k(132)$, because n is a fixed point in $\theta^m(\pi)$ for any m.

But recall that $\theta^{-1}(\alpha)$ is the only element of $\mathcal{T}_n^2(132)$ which contains $(n\ 1)$, so the other cases only allow for $\pi \in \{\alpha_n, \theta^{-1}(\alpha_n), \theta^{-2}(\alpha_n)\}$. It can be shown that for $n \geq 4$, $\alpha_n \in \mathcal{T}_n^3(132) \backslash \mathcal{T}_n^4(132)$, allowing us to establish the desired recurrences.

This concludes the question of enumerating $\mathcal{T}_n^k(\sigma)$ for $\sigma \in \mathcal{S}_3$.

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All figures are my own work and were created with TikZ.

References

[1] Kassie Archer and Robert P. Laudone.
Pattern avoidance and the fundamental bijection, 2024.

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