Hamiltonian cycles in Cayley graphs

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1. Introduction

Definition a ***Hamiltonian* cycle** is a cycle that traverses through each vertex of a graph G exactly once [1].

Definition a ***vertex-transitive graph*** is a graph where its automorphism group is transitive [2]. e.g., a k-cycle is vertex-transitive.

One question relating to Lovász's conjecture asks whether *all finite, connected vertex-transitive graphs have a Hamiltonian cycle* [3]. It held true for most vertex-transitive graphs, there are currently only four known non-trivial cases of vertex-transitive graphs that does not contain a Hamiltonian cycle [3], [4], [5], namely: the Petersen graph, the Coxeter graph, and the two graphs obtained by replacing each vertex in the two graphs above by a triangle (also called the *truncated* graphs of the Petersen and Coxeter graph).

We will discuss the existing proofs of the non-Hamiltonicity of the aforementioned graphs, and conclude its implications in finding Hamiltonian cycles in Cayley graphs.

Definition let G be a group, and $S \subseteq G$ where the identity element $I \notin S$ is the generating set of G. The ***Cayley graph*** $\Gamma(G,\ S)$ is defined as a directed (or undirected) graph having edges $(g,\ gs)$ for any $g \in G$, $s \in S$ [6]. e.g., hypercube graphs are Cayley graphs.

It is also worth noting that all Cayley graphs are vertex-transitive [5].

2. Discussion

The Petersen graph

This proof follows from [7].

Suppose for the contrary that the Petersen graph *is* Hamiltonian. Then, there is a 10-cycle C. Note that the Petersen graph has 15 edges and a smallest cycle of length 5 [7], hence, the graph contains C and 5 chords of C; all chords join vertices along C with a distance at least 5. However, there is no way of drawing 5 such chords without creating a 4-cycle via the drawn chords. Therefore, the Petersen Graph is not Hamiltonian.

The Coxeter graph

First proved by Tutte [8], using a case-by-case approach.

Suppose that there is a Hamiltonian cycle H in the graph. Define the Coxeter graph as follows:

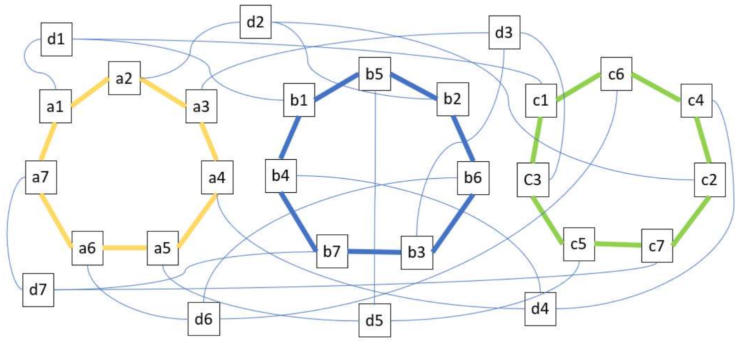

Figure 1: three 7-cycles (left to right) A, B, C, and 7 vertices d_i connecting a_i, b_i and c_i.

We note that the graph is invariant under rotations and reflections, hence we take cycle A for reference; H will not contain all seven edges in A; H will contain at least two consecutive edges of A (e.g., a_1a_2 and a_2a_3 are consecutive edges); and if H contains a_1a_2 and a_2a_3, then H will contain b_2d_2 and d_2c_2 (*).

Let s be the maximum number of consecutive edges of A in H. Notably, $2 \leq s \leq 6$. The proof will explore each of those cases starting from s = 6. If $s = 6$, we let a_1a_2, a_2a_3, a_3a_4, a_5a_6, and a_6a_7 be in H. Branching from the implication (*) we can deduce which other edges are contained by H. However, the deduced graph H consists of 2 cycles, contrary to our assumption. For $s = 5$, we suppose all edges of A except a_1a_7, and a_6a_7 are in H. However, one of these edges must be in H as they are incident to a_7. Thus, the case $s = 5$ is not possible. Case $s = 4$ carries a similar approach and conclusion as $s = 6$, where H will contain 2 cycles. Finally, on the graph after an automorphism defined in [8], cases $s = 2$ and $s = 3$ can be reduced to the case when $s = 4$. Therefore, through all the cases mentioned, the Coxeter graph is not Hamiltonian.

The truncated graphs of the Petersen and Coxeter graph

Definition the ***truncation*** of a graph G, denoted $\mathscr{T}(G)$, is obtained from G by replacing every vertex $v \in V(G)$ by a clique T_v on $deg(v)$ vertices. If $uv \in E(G)$, then one vertex of the clique T_v is adjacent to one vertex of T_u [4].

As discussed in [4], the conclusion where the truncated graphs of the Petersen graph and the Coxeter graph are not Hamiltonian came as a corollary of the following theorem:

Theorem $\mathscr{T}(G)$ of a graph G is Hamiltonian if and only if G has a connected spanning eulerian subgraph.

Definition a **Eulerian cycle** is a closed walk that traverses each edge exactly once [9].

The proof of the theorem will not be included due to length. Following the theorem in the case of 3-regular graphs, if there exists a spanning eulerian subgraph, the subgraph is the Hamiltonian cycle of the graph itself. Hence:

Corollary if G is 3-regular graph, then $\mathscr{T}(G)$ is Hamiltonian if and only if (*iff*) G is Hamiltonian.

As both the Petersen and Coxeter graphs are 3-regular and not Hamiltonian, their truncations are not Hamiltonian.

It is important to note that by the following theorem [4], the truncated graphs are vertex-transitive:

Theorem $\mathscr{T}(G)$ of a graph G is vertex-transitive *iff* G is arc-transitive.

3. Conclusion

While all Cayley graphs are vertex transitive, none of the four graphs mentioned are Cayley [5]. Thus, extending Lovász's conjecture, we were led to the conjecture extensively explored in graph theory [3]:

All Cayley graphs have a Hamiltonian cycle.

4. Future work

All the proofs discussed were exhaustive and we have only proved the non-Hamiltonicity of the graphs, further steps include exploring:

1. Other proofs of non-Hamiltonicity involving group theory
2. Proofs of non-Hamiltonicity using eigenvalues of a graph
3. Proof that the truncation of an arc-transitive graph is vertex-transitive

5. References

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