Sandpile Models for Self-Organised Criticality

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Introduction

Self-Organised Criticality (SOC) is a concept introduced by Per Bak, Chao Tang and Kurt Wiesenfeld (BTW) in 1987 [1] which attempts to explain the complex behavior of critical systems.
A critical state is defined as being a state of a system such that, as a result of the interactions between individual elements, a minor disturbance has the potential to cause events of any scale- including those considered as catastrophic. This phenomenon is commonly encountered in nature, notably moving tectonic plates resulting in earthquakes.
In order to model SOC, BTW introduced the so called *Sandpile Model* [1]. We modify this model to assess the necessary conditions for criticality.

BTW Sandpile Model

Consider an $X \times Y$ dimensional grid. Let the number of grains of sand on cell (i, j) at time step t be $g_t(i, j)$ where $i \in [1, X], j \in [1, Y], t \in \mathbb{N}^0$.

We run each simulation for T timesteps, at each time step t we choose a random position to drop one grain of sand, that is,

$$g_t(i, j) = g_{t-1}(i, j) + 1$$

When number of grains on cell (i, j) reaches a threshold value $c = 4$, the sandpile becomes unstable and a grain of sand from this cell *topples* over to each of it's direct neighbors, that is,

$$g_t(i, j) = g_t(i, j) - 4$$
$$g_t(i \pm 1, j) = g_t(i \pm 1, j) + 1$$
$$g_t(i, j \pm 1) = g_t(i, j \pm 1) + 1$$

If $i \in \{1, X\}$ or $j \in \{1, Y\}$, the grains which would usually topple out of the bounds of the grid are lost from the system.
This model reaches a *critical state* when the number of grains(mass) on the grid stays relatively constant.

Acknowledgements

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SOC in the Model

In a critical state, a *power law distribution* exists between the magnitude of a disturbance to a system, x, and it's frequency, $f(x)$ [1] ie$\exists \alpha, \beta$ st $f(x) \approx \beta x^{\alpha}$.
We plot the frequency distributions on log-log scales of three measures of avalanche size:

- Number of Topples
- Area (no. unique sites which topple)
- Amount of Loss (grains lost from grid)

A mass vs time plot is also produced.

Initial Simulations

When $X = Y$, we see stabilisation of the total mass and evidence of a power law relationship (Figure 1), indicating the critical state is reached.

The deviations from linearity, notably the drop-off at the tail are likely due to the use of a finite sized grid. This is supported by the observation that the larger the board, the better the linear fit as there is higher potential to capture catastrophic events.

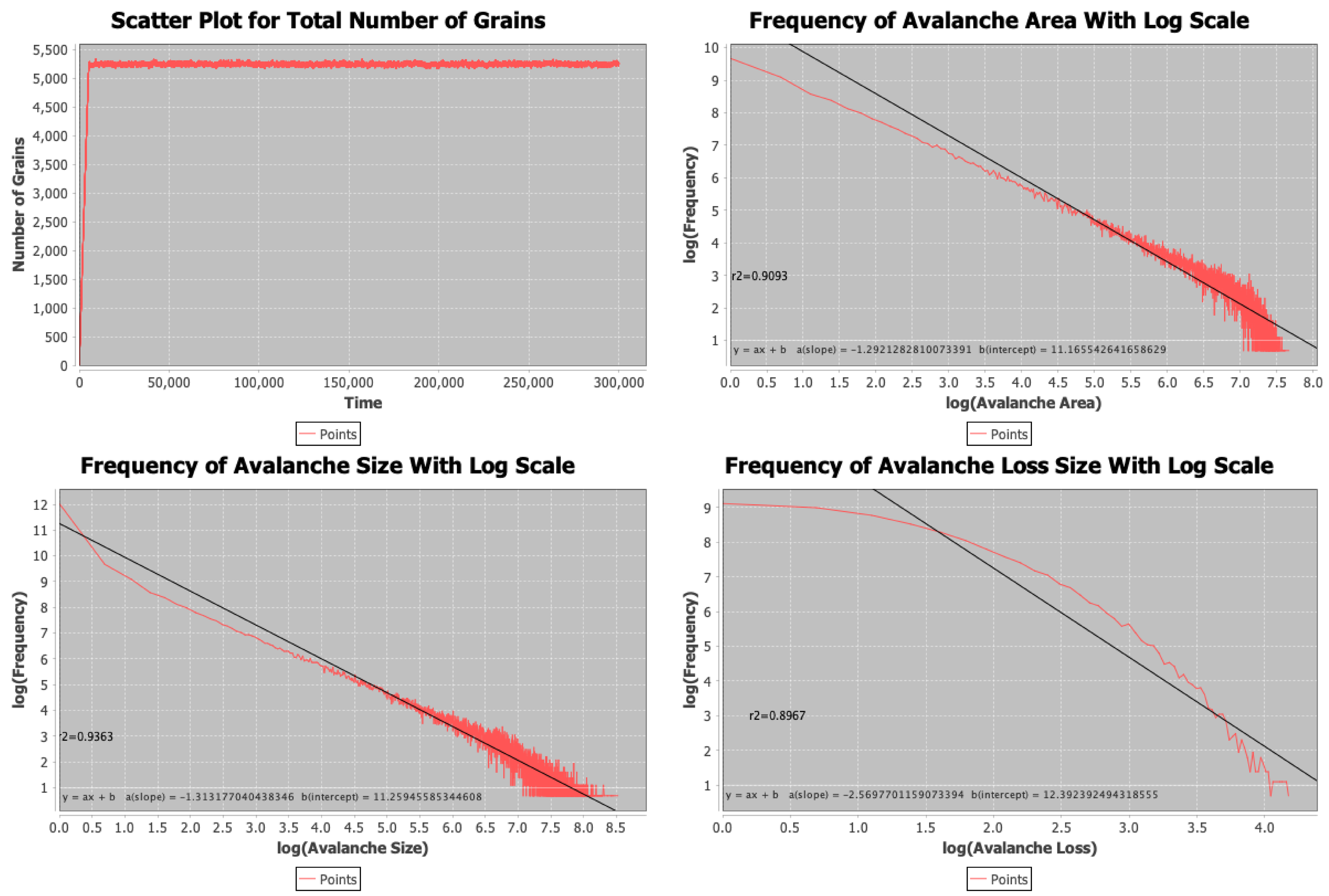

Figure 1: Original BTW model, 50x50 board, T= 300,000

Modifications

Randomness
To test whether the criticality is dependent on randomness, we dropped every grain of sand onto the middle cell of the board and found that the difference in linearity of the frequency graphs is minimal with $R^2 = 0.891$ as opposed to $R^2 = 0.903$.
We also tested the impact of introducing randomness into the system. Instead of a pile automatically toppling when it reaches the threshold c we assigned each cell a probability $p_{i,j}$ of toppling each time a grain is added.

For cells with height, $c \leq h_{i,j} \leq 2c$ and constant $\alpha \in (0, 1)$, let $p_{i,j} = \alpha \cdot (h_{i,j} - c + 1)$.
This did not impact the system's ability to reach a critical state ($R^2 = 0.943$). Thus, we **do not** have evidence that the criticality of the sandpile model is dependent on, nor impaired by randomness.

Wind
We introduced wind into the system to investigate the impact on criticality. Wind may occur in any of the four directions. Instead of one grain falling to the opposite side, two fall in that direction.
Results with equal ($p = 0.25$) probability of breeze in any direction at each topple do not deviate from what is expected of a critical system, however when we introduce the concept of gusts lasting for $\{10, 100, 1000\}$ topples we see less evidence of the power law as shown in Figure 2 where $R^2 = 0.393$.

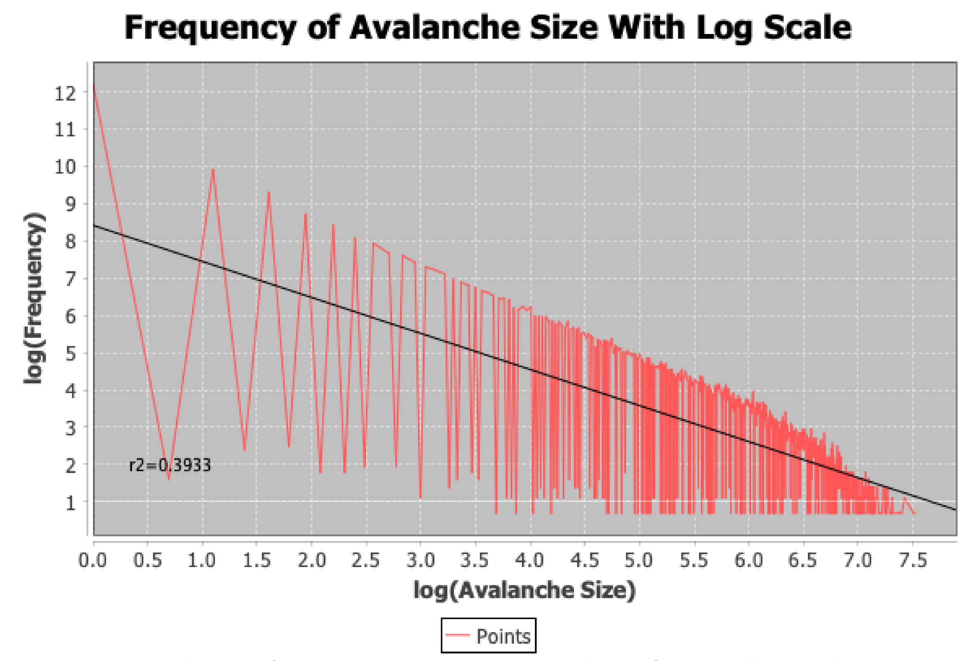

Figure 2: Log-log frequency graph of avalanche topples Gust length 1000, Probability $p = 0.25$

Boundary Conditions

The ability to lose grains is essential to criticality. We assess the necessary amount of output opportunity for the criticality to be unaffected.

We modelled both a annular "board", halving the number of avenues to lose sand grains, and a torus which only allows grains to leave from 2 cells.

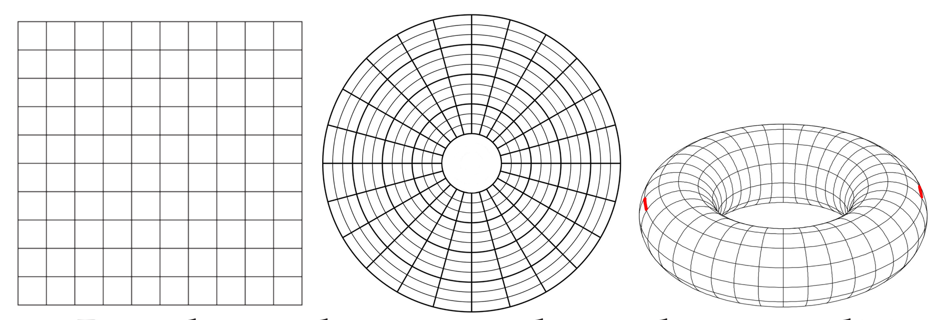

Figure 3: Boards used to assess boundary conditions L to R: Square, annulus & torus with outlet (red)

The mass vs time trajectory to the critical state is very similar. The distinction is the number of topples needed at each time step. For $T = 500$, the total topples are $2948, 3610$ and 8461 respectively as the outlet is reduced.

Our simulations highlight the adaptability of these systems by showing that they can organise themselves into a critical state given there is an avenue for loss, no matter how small that avenue is, emphasising the *self-organising* component of SOC.

References

[1] Per Bak, Chao Tang, and Kurt Wiesenfeld. Self-organized criticality: An explanation of the 1/f noise. *Phys. Rev. Lett.*, 59:381–384, Jul 1987.