Lie group classification and constructions of G_2

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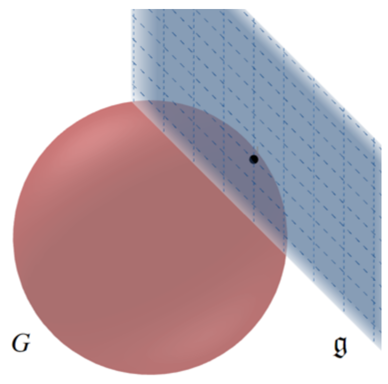
Introduction

Consider a group G which is also a smooth manifold. If the group has a multiplication function ($G \times G \to G$) and an inverse function ($G \to G$) which are smooth, then G is a *Lie Group*

A Lie Group, G, is a *simple Lie group* if it is connected and has no non trivial connected normal subgroups, apart from G itself.



A Lie Group G has a corresponding Lie algebra $\mathfrak{g}$. The Lie Algebra can be thought of as the vector space which is the tangent space at the identity, equipped with a bracket operation $[\cdot\,,\,\cdot]$ satisfying the Jacobi Identity:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] \;=\; 0$$

In the case of a matrix group, the bracket operation is the commutator: $[X,Y] = XY - YX$

Should one attempt to classify the complex simple Lie algebras, and thence the compact simple Lie groups, one might quickly stumble across 4 infinite families of simple classical groups.

Yet the geometric nature of Dynkin Diagrams suggests the possibility of exactly 5 more 'exceptional' Lie Groups. The onus has now been bestowed upon the inquirer: find explicit examples of these exceptional groups, or show that they cannot exist. We will investigate the smallest of these groups, the 14-dimensional Lie Group G_2

1. The Classification of Lie Groups

The classification of compact simple Lie Groups generally goes as such: We first find the roots:

- Find a maximal torus, T, of the Lie group, that is, a maximal connected and compact abelian subgroup of G.
- Find the complexification of the Lie algebra of T: $\mathfrak{t}_{\mathbb{C}}$
- Find the roots, that is, the α in the dual space of $\mathfrak{t}$ with:

$$\exists z \neq 0 \in \mathfrak{g}_{\mathbb{C}} \text{ such that } [x,z] = i\alpha(x)z, \forall x \in \mathfrak{t}$$

- The Lie algebra $\mathfrak{g}$ then decomposes as: $\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \bigoplus_{\alpha} \mathfrak{g}_{\alpha}$

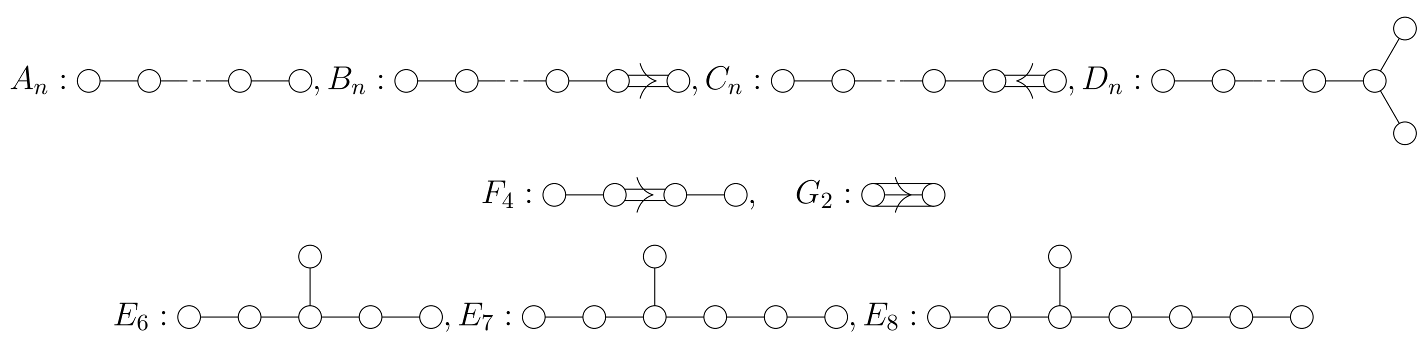
  where $\mathfrak{g}_\alpha = \{z \in \mathfrak{g} | x \in \mathfrak{t} \implies [x,z] = i\alpha(x)z\}$ are one dimensional root spaces.

We choose a set of simple roots, and then can then use an inner product (derived from the Killing Form) to convert these roots into a root system. Such an inner product allows us to determine the length of the roots and angles between them, giving us a root diagram.

Under such an inner product, the angles between roots are very constrained. In particular, only the angles of $\frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{6}, \frac{5\pi}{6}$ are possible.

The corresponding *Dynkin Diagram* is made by assigning each simple root and node, and connecting nodes with edges based on the angle between them. However, such diagrams are very restricted.

Theorem: there are 4 infinite families of connected, irreducible Dynkin Diagrams, and 5 exceptional ones, namely:



The Dynkin Diagrams for A_n, B_n, C_n, D_n correspond to the matrix groups $SU(n+1)$, $SO(2n+1)$, $Sp(n)$ and $SO(n)$ respectively.

Theorem: the Dynkin Diagrams and complex simple Lie algebras determine each other precisely up to isomorphism.

For a complex simple Lie algebra, there are real forms, whose complexification is the complex algebra. Two of such are the *compact real form*, corresponding to compact Lie groups, and the *split real form*, whose Lie groups are non-compact. The map from real Lie groups to their corresponding real Lie algebra is many-to-one.

2. An example: $SU(3)$

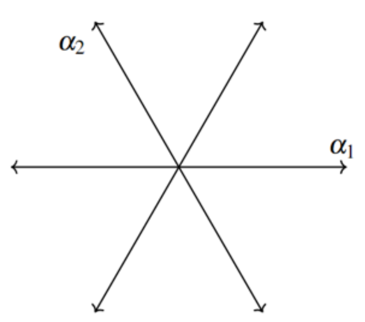
We will find the root diagram and Dynkin Diagram for $G = SU(3)$. The complexified Lie algebra is $\mathfrak{g}_{\mathbb{C}} = \mathfrak{su}(3)_{\mathbb{C}} = \mathfrak{sl}(3,\mathbb{C})$, the set of 3×3 traceless matrices.

Note that $SU(3)$ has the maximal torus $\{diag(e^{i\theta_1}, e^{i\theta_2}, e^{-i\theta_1 - i\theta_2}), \theta_1, \theta_2 \in \mathbb{R}\}$ hence $H_1 = diag(i,0,-i), H_2 = diag(0,i,-i)$ is a basis for $\mathfrak{t}$.

H_1, H_2 along with matrices $E_{12}, E_{13}, E_{23}, E_{21}, E_{31}, E_{32}$ form a basis for $\mathfrak{sl}(3,\mathbb{C})$. We then have the following calculation:

$$[sH_1 + tH_2, E_{12}] = (2s-t)iE_{12}$$
$$[sH_1 + tH_2, E_{13}] = (s+t)iE_{13}$$
$$[sH_1 + tH_2, E_{23}] = (-s+2t)iE_{23}$$

Hence our roots are $\pm\alpha = 2s - t, s + t, -s + 2t$.
We take the roots $\alpha_1 = 2s - t$ and $\alpha_2 = -s + 2t$ as simple roots, since they generate all the roots using only positive or only negative integer coefficients.



The inner product tells us that all our roots are the same length, and are at $60°$ from each other, shown on left.

Moreover, the simple roots are at $120°$, so the Dynkin diagram tells us to join the nodes using one line:

$A_2 :$ ○—○

References and Acknowledgements

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3. The Octonions and Split Octonions

The division algebras: real numbers, complex numbers, quaternions and octonions (and their split varieties) can all be constructed from the real numbers via the Cayley Dickson construction:

We can define the complex numbers as $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}$ with multiplication and conjugation as

$$(a,b)(c,d) = (ac - bd, ad + bc), \quad \overline{(a,b)} = (a,-b)$$

The complex numbers are an associative and commutative division algebra. We more commonly write $(a,b) = a + bi$ where $i^2 = -1$

We can define the quaternions as $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}$ with multiplication and conjugation as

$$(a,b)(c,d) = (ac - \overline{d}b, da + b\overline{c}), \quad \overline{(a,b)} = (\overline{a}, -b)$$

The quaternions are an associative but not commutative division algebra. We more commonly write $(a+bi, c+di) = a + bi + cj + dk$ where $i^2 = j^2 = k^2 = ijk = -1$

We can then define the octonions $\mathbb{O}$ and split octonions $\mathbb{O}'$, both as $\mathbb{H} \oplus \mathbb{H}$. For the octonions, multiplication is defined the same as for the quaternions: $(a,b)(c,d) = (ac - \overline{d}b, da + b\overline{c})$. However, for the split octonions, multiplication is defined as $(a,b)(c,d) = (ac + \overline{d}b, da + b\overline{c})$.

Neither the octonions nor split octonions are associative nor commutative, hence cannot be represented by matrices. However, the octonions (but not the split octonions) are still a division algebra.

Just as the quaternions introduce 2 more orthonormal vectors to the complex numbers, both which square to -1, the octonions add 4 more orthonormal vectors to the quaternions which all square to -1. However, the split octonions add 4 orthonormal vectors which square to 1.

4. Cartan's Construction of G_2

It was Elie Cartan and Wilhelm Killing who were the first to classify the simple Lie Groups. In 1908, Cartan claimed that G_2 is the automorphism group of the Octonions, $Aut(\mathbb{O})$.

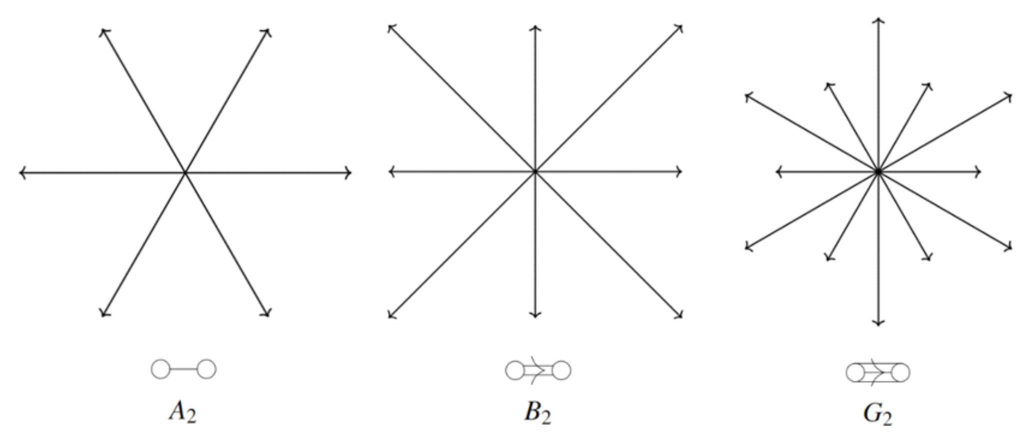
Any (continuous) automorphism group must preserve the real numbers, $\mathbb{R}$. It can be shown that the Automorphism group of the octonions is a subgroup of $SO(7)$ which fixes $\mathbb{R}$ and preserves the 7-dimensional cross product on the imaginary octonions (those without a real part): $x \times y = \frac{1}{2}(xy - yx)$

Theorem: For both the octonions and split octonions, the automorphism group is a 14 dimensional simple Lie group.

Theorem: Both automorphism groups have 2 dimensional maximal tori, so we say they are of *rank 2*.

Since the Lie Algebra for a Lie group can be decomposed into $\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \bigoplus_{\alpha} \mathfrak{g}_\alpha$, we have exactly $14 - 2 = 12$ root spaces, corresponding to 12 roots.

There are only 3 possibilities for root diagrams from the rank 2 Dynkin Diagrams (those with 2 dots):

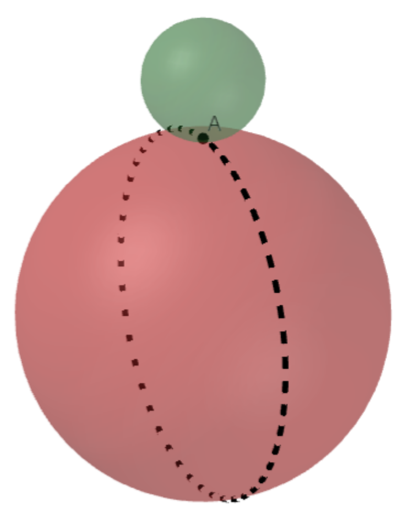


One might notice that the root diagram for A_2 is embedded in that for G_2. This is no coincidence: $SU(3)$ is the subgroup of G_2 which preserves the unit vector i

Only G_2 has a root diagram with 12 roots. We arrive at the following:

Theorem: The automorphism groups $Aut(\mathbb{O})$ and $Aut(\mathbb{O}')$ are of type G_2. Moreover, the automorphism group of the octonions is the compact real form of G_2 whereas the automorphism group of the split octonions is the split real form of $G_2 : G_2'$

5. A geometric G_2



We consider a 'rolling' ball of radius 1, rolling atop a 'fixed' ball of radius R>1, without twisting or slipping. We define a 'point' in this space as any configuration of the balls. The rolling ball might be in any rotation of $SO(3)$, and can be in contact with any point of the 'fixed' ball. Hence the set of points is $SO(3) \times S^2$.

We can then define 'lines' in this space as the subset of points, $L \subset SO(3) \times S^2$ that a ball with some starting configuration takes when rolling along a great arc of S^2.

If R=3, the symmetry group of this geometry is *almost* G_2. To get G_2 we need to make a few adjustments.

The spin group, $Spin(n)$ is a double cover of $SO(n)$. In the case $n = 3$, $Spin(3)$ is isomorphic to $SU(2)$, that is, there is a double cover of $SU(2)$ onto $SO(3)$.

The real projective plane, $\mathbb{RP}^2$, is the set of all 1 dimensional subspaces of $\mathbb{R}^3$. Each $x \in \mathbb{RP}^2$ can be associated with the two points in which the 1 dimensional subspace intersects the sphere S^2. In this sense, we can think of $\mathbb{RP}^2$ as the unit 2-sphere, where antipodal points are the same.

We now consider a spinor, that is, a ball that must roll twice to return to its original state, rolling atop the real projective plane; a sphere whose antipodal points are the 'same'.

The spinor can be in any configuration of $SU(2)$, and can be anywhere on the real projective plane, $\mathbb{RP}^2$. The set of all points is then $SU(2) \times \mathbb{RP}^2$. The lines are then the subsets of configurations that are obtained when the spinor rolls along a great arc of $\mathbb{RP}^2$.

We consider a second geometry. We define a *null subalgebra* of the split octonions, $\mathbb{O}'$, to be a vector subspace V on which the product vanishes. That is, if $x, y \in V$ then $xy = 0$.

In this geometry, the 'points' are the 1 dimensional null subalgebras. Our 'lines' are exactly the 2 dimensional null subalgebras.

Note that both geometries are examples of *incidence geometries*, that is, a collection of objects: points and lines, and an incidence relation: a notion of a point lying on a line. We then have the following:

Theorem:: (Baez and Huerta, 2014). If and only if R=3, these two geometries are equivalent. That is, there is an isomorphism between points and lines in the rolling ball and octonion geometries.

An automorphism of the split octonions will preserve null subspaces. Hence, when R=3, the symmetry group of the rolling ball geometry, that is, the symmetry group that maps lines to lines, is precisely G_2'