Polynomial Division Primer

John Banks and Liz Bailey

- We can divide polynomials using the same techniques of long division as we use with natural numbers – but we get polynomials as our quotient and remainder.
- We first explain how this works before showing the quick tabular way to do the calculations.
- This slide presentation is best viewed on screen in full screen mode.
- Just view one slide at a time and try to understand what is happening before moving on.

What are we trying to do?

- In the following examples, we divide the polynomial $p(x)$ by the polynomial $d(x)$, obtaining an answer in the form

$$\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.$$

  where $r(x)$ has (strictly) smaller degree than $d(x)$.
- This is achieved by repeatedly subtracting appropriate multiples of $d(x)$.
- We choose these multiples so that the "leading" term is removed from $p(x)$ by each subtraction.
- We can do this most efficiently in a tabular format, but first lets see why it works.

First Example: Step 1

$$p(x) = x^4 + 3x^3 - 3x + 5, \quad d(x) = x + 2.$$

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so $\dfrac{p(x)}{d(x)}$ is x^3 plus a fraction $\dfrac{p_1(x)}{d(x)}$ with numerator $p_1(x) = x^3 - 3x + 5$ of smaller degree than $p(x)$.

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  so $\frac{p(x)}{d(x)}$ is x^3 plus a fraction $\frac{p_1(x)}{d(x)}$ with numerator $p_1(x) = x^3 - 3x + 5$ of smaller degree than $p(x)$.
- We can do exactly the same thing to this new fraction!

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  so $\frac{p(x)}{d(x)}$ is $x^3 + x^2$ plus a fraction $\frac{p_2(x)}{d(x)}$ with numerator $p_2(x) = -2x^2 - 3x + 5$ of even smaller degree.

- So what do we do next ...

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- … remove the leading term $-2x^2$ of $p_2(x)$, by subtracting $(-2x)d(x)$ of course!

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&\Rightarrow \frac{p(x)}{d(x)} = x^3 + x^2 - 2x + \frac{x+5}{d(x)}
\end{aligned}$$

...so $\dfrac{p(x)}{d(x)}$ is $x^3 + x^2 - 2x$ plus a fraction in which the numerator $p_3(x) = x + 5$ has even smaller degree.

- This is so much fun. Lets do it again!

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- To remove the leading term x of $p_3(x)$, subtract $1 \times d(x)$:

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\Rightarrow \frac{p(x)}{d(x)} = x^3 + x^2 - 2x + 1 + \frac{3}{d(x)}
\end{aligned}
$$

  ...expressing $\frac{p(x)}{d(x)}$ as a polynomial $x^3 + x^2 - 2x + 1$ plus a fraction in which the numerator $p_4(x) = 3$ has even smaller degree.

- Can we do this again?

First Example: The end!

- No! The leading term 3 of $p_4(x) = 3$ is now too small (in degree) to express as a multiple of $d(x)$.

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$$\frac{p(x)}{d(x)} = x^3 + x^2 - 2x + 1 + \frac{3}{x+2}$$

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- Our job is done!
- That was fun, but involved a lot of writing.
- We can carry out the division much more efficiently in a tabular format.

Same Example in Tabular Format

$$x+2\overline{)\,x^4+3x^3+0x^2-\ 3x+5}$$

Same Example in Tabular Format

$$x+2\,\overline{)\,x^4+3x^3+0x^2-\ 3x+5}$$

Easier to maintain alignment
if we add any “missing powers”

Same Example in Tabular Format

$$x+2\overline{)\,x^4+3x^3+0x^2-\ 3x+5}$$

Need a product of $x+2$ that removes x^4 . . .

Same Example in Tabular Format

$$\begin{array}{r} x^3 \\ x+2\overline{)\,x^4 + 3x^3 + 0x^2 - 3x + 5} \end{array}$$

…so multiply by x^3.

Same Example in Tabular Format

$$
\begin{array}{r@{}l}
 & \quad\quad x^3 \\ \cline{2-2}
x+2\,) & x^4 + 3x^3 + 0x^2 - 3x + 5 \\
 & \underline{x^4 + 2x^3}
\end{array}
$$

$$x^3(x+2) = x^4 + 2x^3.$$

Same Example in Tabular Format

$$
\begin{array}{r}
x^3 \\
x+2 \overline{) x^4 + 3x^3 + 0x^2 - 3x + 5} \\
\underline{x^4 + 2x^3} \\
x^3 + 0x^2 - 3x + 5
\end{array}
$$

Subtracting gives

$x^3 + 0x^2 - 3x + 5$.

Same Example in Tabular Format

$$
\begin{array}{r|l}
 & x^3 \\
\cline{2-2}
x+2 & x^4 + 3x^3 + 0x^2 - 3x + 5 \\
 & \underline{x^4 + 2x^3} \\
 & \qquad x^3 + 0x^2 - 3x + 5
\end{array}
$$

Need a product of $x + 2$ that removes x^3 . . .

Same Example in Tabular Format

$$
\begin{array}{r|l}
 & x^3 + x^2 \\
\hline
x + 2 & x^4 + 3x^3 + 0x^2 - 3x + 5 \\
 & \underline{x^4 + 2x^3} \\
 & \quad\quad x^3 + 0x^2 - 3x + 5
\end{array}
$$

...so multiply by x^2.

Same Example in Tabular Format

$$
\begin{array}{rl}
 & x^3 + x^2 \\
x + 2 \,\big) & \overline{x^4 + 3x^3 + 0x^2 - 3x + 5} \\
 & \underline{x^4 + 2x^3} \\
 & \qquad\quad x^3 + 0x^2 - 3x + 5 \\
 & \qquad\quad \underline{x^3 + 2x^2}
\end{array}
$$

$$x^2(x + 2) = x^3 + 2x^2.$$

Same Example in Tabular Format

$$
\begin{array}{r}
x^3 + x^2 \phantom{{}- 3x + 5} \\
x+2 \overline{\big)\, x^4 + 3x^3 + 0x^2 - 3x + 5} \\
\underline{x^4 + 2x^3} \phantom{{}+ 0x^2 - 3x + 5} \\
x^3 + 0x^2 - 3x + 5 \\
\underline{x^3 + 2x^2} \phantom{{}- 3x + 5} \\
-2x^2 - 3x + 5
\end{array}
$$

Subtracting gives $-2x^2 - 3x + 5$.

Same Example in Tabular Format

$$
\begin{array}{r}
x^3 + x^2 \phantom{{}+0x^2 - 3x + 5} \\
x+2 \overline{) x^4 + 3x^3 + 0x^2 - 3x + 5} \\
\underline{x^4 + 2x^3} \phantom{{}+ 0x^2 - 3x + 5} \\
x^3 + 0x^2 - 3x + 5 \\
\underline{x^3 + 2x^2} \phantom{{}- 3x + 5} \\
-2x^2 - 3x + 5
\end{array}
$$

Need a product of $x + 2$ that removes $-2x^2$ . . .

Same Example in Tabular Format

$$
\begin{array}{r}
x^3 + x^2 - 2x \\
x+2 \overline{) x^4 + 3x^3 + 0x^2 - 3x + 5} \\
\underline{x^4 + 2x^3} \\
x^3 + 0x^2 - 3x + 5 \\
\underline{x^3 + 2x^2} \\
-2x^2 - 3x + 5
\end{array}
$$

... so multiply by $-2x$.

Same Example in Tabular Format

$$
\begin{array}{r}
x^3 + x^2 - 2x \\
x+2\overline{\big)\, x^4 + 3x^3 + 0x^2 - 3x + 5} \\
\underline{x^4 + 2x^3} \\
x^3 + 0x^2 - 3x + 5 \\
\underline{x^3 + 2x^2} \\
-2x^2 - 3x + 5 \\
\underline{-2x^2 - 4x}
\end{array}
$$

$$-2x(x+2) = -2x^2 - 4x.$$

Same Example in Tabular Format

$$
\begin{array}{r}
x^3 + x^2 - 2x \phantom{{}+5} \\
x+2\overline{\smash{)}\, x^4 + 3x^3 + 0x^2 - 3x + 5} \\
\underline{x^4 + 2x^3} \phantom{{}+0x^2 - 3x + 5} \\
x^3 + 0x^2 - 3x + 5 \\
\underline{x^3 + 2x^2} \phantom{{}- 3x + 5} \\
-2x^2 - 3x + 5 \\
\underline{-2x^2 - 4x} \phantom{{}+5} \\
x + 5
\end{array}
$$

Subtracting gives $x + 5$.

Same Example in Tabular Format

$$
\begin{array}{r}
x^3 + x^2 - 2x \\
x+2 \overline{\smash{)}\, x^4 + 3x^3 + 0x^2 - 3x + 5} \\
\underline{x^4 + 2x^3} \\
x^3 + 0x^2 - 3x + 5 \\
\underline{x^3 + 2x^2} \\
-2x^2 - 3x + 5 \\
\underline{-2x^2 - 4x} \\
x + 5
\end{array}
$$

Need a product of $x + 2$ that removes x ...

Same Example in Tabular Format

$$
\begin{array}{r}
x^3 + x^2 - 2x + 1 \\
x+2\overline{)\,x^4 + 3x^3 + 0x^2 - 3x + 5} \\
\underline{x^4 + 2x^3} \qquad\qquad\qquad\quad \\
x^3 + 0x^2 - 3x + 5 \\
\underline{x^3 + 2x^2} \qquad\qquad \\
-2x^2 - 3x + 5 \\
\underline{-2x^2 - 4x} \quad\;\; \\
x + 5
\end{array}
$$

. . . so multiply by 1.

Same Example in Tabular Format

$$
\begin{array}{r}
x^3 + x^2 - 2x + 1 \\
x+2 \overline{) x^4 + 3x^3 + 0x^2 - 3x + 5} \\
\underline{x^4 + 2x^3} \phantom{+ 0x^2 - 3x + 5} \\
x^3 + 0x^2 - 3x + 5 \\
\underline{x^3 + 2x^2} \\
-2x^2 - 3x + 5 \\
\underline{-2x^2 - 4x} \\
x + 5 \\
\underline{x + 2}
\end{array}
\qquad 1 \times (x+2) = x+2.
$$

Same Example in Tabular Format

$$
\begin{array}{r}
x^3 + x^2 - 2x + 1 \\
x+2 \,\overline{\smash{)}\, x^4 + 3x^3 + 0x^2 - 3x + 5} \\
\underline{x^4 + 2x^3} \\
x^3 + 0x^2 - 3x + 5 \\
\underline{x^3 + 2x^2} \\
-2x^2 - 3x + 5 \\
\underline{-2x^2 - 4x} \\
x + 5 \\
\underline{x + 2} \\
3
\end{array}
$$

Subtracting gives 3 and we stop.

Same Example in Tabular Format

$$
\begin{array}{r}
x^3 + x^2 - 2x + 1 \\
x+2\,\overline{)\,x^4 + 3x^3 + 0x^2 - 3x + 5} \\
\underline{x^4 + 2x^3} \qquad\qquad\qquad \\
x^3 + 0x^2 - 3x + 5 \\
\underline{x^3 + 2x^2} \qquad\quad\;\; \\
-2x^2 - 3x + 5 \\
\underline{-2x^2 - 4x} \quad\;\; \\
x + 5 \\
\underline{x + 2} \\
3
\end{array}
$$

Answer is:

polynomial plus remainder

$$\frac{p(x)}{d(x)} = x^3 + x^2 - 2x + 1 + \frac{3}{x+2}$$

Same Example in Tabular Format

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\begin{array}{r}
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-2x^2 - 3x + 5 \\
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x + 5 \\
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$$

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x^3 + \ \ x^2 - \ 2x + 1 \\
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\underline{x^4 + 2x^3} \qquad\qquad\qquad\quad \\
x^3 + 0x^2 \color{red}{- 3x + 5} \\
\underline{x^3 + 2x^2} \qquad\qquad \\
- 2x^2 - \ 3x \color{red}{+ 5} \\
\underline{-2x^2 - 4x} \quad\ \ \\
x + 5 \\
\underline{x + 2} \\
3
\end{array}
$$

Answer is:

polynomial plus remainder

$$\frac{p(x)}{d(x)} = x^3 + x^2 - 2x + 1 + \frac{3}{x+2}$$

When we subtract, we can leave out unnecessary terms.

Another example

- When we subtract, we sometimes get lucky!

$$p(x) = 8x^3 - 4x^2 - 2x + 1, \quad d(x) = 2x - 1.$$

$$2x - 1 \overline{) \, 8x^3 - 4x^2 - \; 2x + 1}$$

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$$\begin{array}{r|l} & 4x^2 \\ \hline 2x - 1 & 8x^3 - 4x^2 - 2x + 1 \\ & \underline{8x^3 - 4x^2} \\ & \quad\quad 0 \quad - 2x + 1 \end{array}$$

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$$p(x) = 8x^3 - 4x^2 - 2x + 1, \quad d(x) = 2x - 1.$$

$$\begin{array}{r}
4x^2 \quad - \quad 1 \\
2x - 1 \overline{) \, 8x^3 - 4x^2 - 2x + 1} \\
\underline{8x^3 - 4x^2} \qquad\qquad \\
0 \quad - 2x + 1 \quad
\end{array}$$

Another example

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 & \; 4x^2 \quad - \quad 1 \\
\cline{2-2}
2x - 1 & \big) \; 8x^3 - 4x^2 - \; 2x + 1 \\
 & \;\; \underline{8x^3 - 4x^2} \\
 & \qquad\quad 0 \; - 2x + 1 \\
 & \qquad\qquad\; \underline{-2x + 1} \\
 & \qquad\qquad\qquad\;\; 0
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\underline{8x^3 - 4x^2} \qquad\qquad \\
0 \quad - 2x + 1 \\
\underline{-2x + 1} \\
0
\end{array}$$

Answer:

$$\frac{p(x)}{d(x)} = 4x^2 - 1 + \frac{0}{2x - 1}$$
$$= 4x^2 - 1$$