# The moduli space of spatial polygons

Enda Han supervised by Paul Norbury



#### Introduction

We first try to introduce the notion of a polygon. We say that a polygon is a collection of vertices connected by a collection of edges in some cyclic order such that they begin and end at the same point. As such we allow for intersecting edges and even vertices with more than 2 coinciding edges (see Figure 1).

Fig. 1: Examples of polygons in the plane

Now finally we lift these polygons from Euclidean 2space into Euclidean 3-space and identify polygons that we equivalent up to isometry. We have thus constructed our spatial polygons.

Of interest to us is the space of polygons with fixed edge lengths (but not necessarily direction).

Here consider fixing side lengths  $r_1, r_2, r_3, r_4$  in Figure 3 and allowing  $l_1, l_2$  to vary. One might be able to see the space of polygons we refer to appears and indeed has dimension 2 which we call a moduli space.

Important is that this moduli space is a smooth manifold of dimension 2(n-3)[3] iff there does not exist a polygon that degenerates to a line in the space. More precisely, there exists no subset of  $1, 2, \ldots, n$  *I* such that

$$\sum_{I} r_i - \sum_{I^c} r_i = 0$$

which allows for a polygon where all edges are parallel to one another. We define  $M'(r)\subset M(r)$  as the smooth manifold resulting from the removal of all such degenerate polygons.

### Symplectic Volume

in some reasonable order (see Figure 2) and the amount that we bend at each diagonal  $\theta_1, \theta_2, \ldots, \theta_{n-3}$ .

**Theorem 2 (Kapovich-Millson)** *The volume form* on M'(r) *is* 

 $dl_1d\theta_1\ldots dl_{n-3}d\theta_{n-3}$ 

**Corollary 2.1** Let  $f_1 = min(r_1 + \cdots + r_{n-2}, r_{n-1} + r_n)$ and  $f_2 = max(min(| \pm r_1 \pm \cdots \pm r_{n-2}|, |r_{n-1} - r_n|)$ Symplectic volume of M'(r) is given by

vol 
$$M'(r) = 2\pi \int_{f_2}^{f_1} vol(M((r_1, \dots, r_{n-2}, l)) dl)$$

#### n = 4

An attempt was made to geometrically prove the Mandini formula from our given volume form. Here we prove the n = 4 case.

We first note that our volume in the n = 4 case is as follows (recall that there is only one diagonal):

$$\int_{0}^{2\pi} \int_{f_2}^{f_1} dl d\theta = 2\pi \int_{f_2}^{f_2} dl = 2\pi (f_1 - f_2)$$

Define  $\varepsilon_{I \text{long}} = |\varepsilon_I|$ . We now realise that

$$\begin{split} \min(r_1 + r_2, r_3 + r_4) &= \frac{1}{2} (|\varepsilon_{\{1,2,3,4\}}| - |\varepsilon_{\{1,2\}}|) \\ \max(|r_1 - r_2|, |r_3 - r_4|) &= \frac{1}{2} (|\varepsilon_{\{1,3\}}| + |\varepsilon_{\{1,4\}}|) \end{split}$$

It thus follows that:

$$\begin{array}{l} \operatorname{vol} M'(r) = \pi(|(\varepsilon_{\{1,2,3,4\}}| - |\varepsilon_{\{1,2\}}|) - (|\varepsilon_{\{1,3\}}| + |\varepsilon_{\{1,4\}}|)) \\ = \pi(|\varepsilon_{\{1,2,3\}}| + |\varepsilon_{\{1,2,4\}}| + |\varepsilon_{\{2,3,4\}}| - |\varepsilon_{\{1,2,3,4\}}| \\ - |\varepsilon_{\{1,2\}}| - |\varepsilon_{\{1,3\}}| - |\varepsilon_{\{1,4\}}|) \end{array}$$

# **Permuting side lengths**

An interesting corollary that follows from the Mandini formula is that switching the order of side lengths, although effecting the polygons formed, will not change the symplectic volume of the resulting manifold. Here, we show that it is easily derived from 2.1. We first note that it suffices to show that switching the n - 1th and nth sides will leave volume invariant. This is true as any set of diagonals  $l_i$  along with their angles result in a volume form. In this way, we may always permute consecutive sides with a wise choice of vertex through which our diagonals pass. We note that  $S_n$  is generated by 2-cycles which implies we may achieve any desired permutation. We now recall our explicit volume formula:

$$2\pi \int_{f_2}^{f_1} \operatorname{vol}(M((r_1, \dots, r_{n-2}, l)) \, dl$$

and immediately realise that switching  $r_{n-1}$  and  $r_n$  have no affect on our boundary conditions (as  $r_{n-1} + r_n = r_n + r_{n-1}$  and  $|r_{n-1} - r_n| = |r_n - r_{n-1}|$ ) nor are they present in our integrand. We are done.

# The relationship between a partial derivative and n-1

Finally, it was shown that there is a relationship between differentiation and reduction in the number of sides of a polygon. Put explicitly:

$$\frac{\partial}{\partial r_n}\Big|_{r_n=0} \left( \operatorname{vol} M(r_1,\ldots,r_n) \right) = 4\pi \operatorname{vol} M(r_1,\ldots,r_{n-1})$$

Which was motivated by Corollary 2.1 and proved using Theorem 1.

#### **Symplectic Volume**

Of interest to us is some notion of volume over these moduli spaces which we call its symplectic volume. The calculation of such volumes is the topic of this project. One might be able to visualise these volumes for n = 4. Considering Figure 3 we see that diagonal  $l_1$  and an angle on this edge characterises M(r). Indeed it can be shown that M(r) is exactly a 2-sphere [3] and the area of said 2-sphere is an example of symplectic volume. Important are two results from prior literature [2][1]. Define  $\varepsilon_I(r) = \sum_I r_i - \sum_{I^e} r_i$  We call I long if

 $\varepsilon_I(r) > 0$ 

**Theorem 1 (Mandini)** Symplectic volume of M'(r) is given by the piecewise polynomial function:

vol 
$$M'(r) = \frac{(2\pi)^{n-3}}{2(n-3)!} \sum_{I \text{ long}} (-1)^{n-|I|} \varepsilon_I(r)^{n-3}$$

We note that all of our polygons may be constructed by pasting together n - 2 triangles all of which share a common vertex  $v_i$  and allowing for our polygons to be 'bent' about the edges of the triangle. We label all n - 3non trivial diagonals intersecting with  $v_i \ l_1, l_2, \ldots, l_{n-3}$ 



Fig. 2: A labelling of *ls* 

 $= -\frac{2\pi}{2} \sum_{I = 1} (-1)^{n-|I|} \varepsilon_I(r)$ 

As desired.

#### On the choice of coordinates

A result of the Kapovich-Millson volume form result is the existence of several different volume forms. More specifically, we may take any vertex as our common vertex resulting in several different volume forms (see Figure 3). Indeed it must follows that the determinant of the change of coordinates Jacobian between these coordinate systems must be 1 up to a sign which is non-trivial.



Fig. 3: A quadrilateral with both possible diagonals drawn in forming a tetrahedron

The following theorem was proved inductively.

**Theorem 3** Given our desired Jacobian has determinant 1 (up to sign) in the the n = 4 case, any choice of appropriate coordinates for  $n \ge 5$  will also have Jacobian determinant 1.

# Conclusion

In this project, the symplectic volume of spatial polygons, more particularly the relationship between the explicit Mandini Formula and our derived integral form for the symplectic volume using the volume form described by Kapovich and Millson, was explored. An elementary proof for the unification of these two formulae in the n = 4 case was found and the presence of several volume forms was exploited when seeking to prove qualities immediately obvious in one formulaic representation but not in the other. Perhaps of most interest and certainly worthy of further exploration is the relationship between the volume of polygons in higher dimensions. A derivative relationship was explicitly found to link n-gons to (n-1)-gons which might prove helpful in revealing more of the geometric structure of these volumes and serve in finding an elementary proof unifying the two formulae described for arbitrary n.

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