# THE SEIFERT-VAN KAMPEN THEOREM

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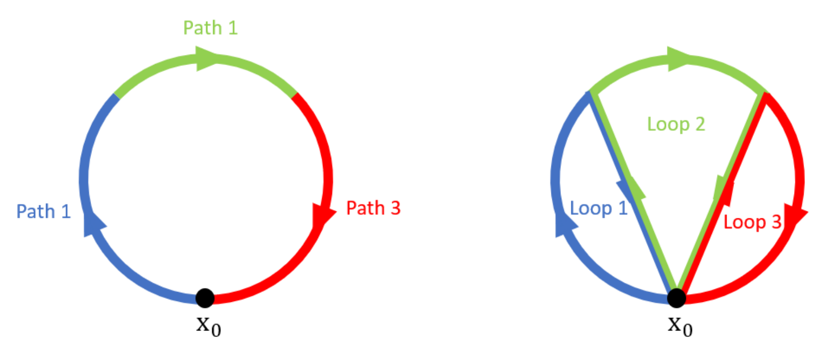
## Introduction

The fundamental group is a powerful tool in the study of topological spaces. A natural question arises when we consider the union of spaces: how do we deduce the fundamental group of the union space from the fundamental groups of individual spaces? The Seifert-van Kampen theorem answers the question partially. In particular, if $C$ is an open cover of the space $X$ that is closed under finite intersection, such that each $U \in C$ is path-connected and contains the basepoint $x_0 \in X$, then we can write $\pi_1(X, x_0)$ in terms of the $\pi_1(U, x_0)$'s.

## The Fundamental Group Version

Next we try to apply the subdivision trick to fundamental groups to obtain $\pi_1(X, x_0) = colim_{U\in C}\pi_1(U, x_0)$. Here we have a problem: once we subdivide a loop, the subdivisions are no longer loops, but mere paths(2.1). In order to turn them into loops, we have to choose a path from $x_0$ to each of the end points(2.2). Further more, we want these paths to be in the same elements of C as the corresponding subdivisions, so that every resulting loop also lies in that element.




2.1 2.2

So for each $U \in C$ we are choosing a path $f_y^U$ from $x_0$ to every $y \in U$. Categorically, this choice corresponds to a functor
$F_U : \Pi(U) \to \pi_1(U, x_0)$ which sends a morphism $[g] : y \to z$ to $[f_y^U * g * \overline{f_z^U}]$.
Let $J_U : \pi_1(U, x_0) \to \Pi(U)$ denote the inclusion functor, then since C is finite, we can choose the $f_y^U$'s such that
1. For any $U \in C$, $f_{x_0}^U$ is the constant path at $x_0$
2. For any $U, V \in C$, $f_y^U = f_y^V$ if $y \in U \cap V$
$F := \{F_U\}_{U\in C} : \Pi|_{\mathscr{C}} \to \pi_1|_{\mathscr{C}}$ is a natural transformation with $F \circ J = Id$.
For every $y \in X$, let $f_y^X = f_y^U$ if $y \in U$. Since C covers X, the $f_y^X$'s define a functor $F_X : \Pi(X) \to \pi_1(X, x_0)$ with $F_X \circ J_X = Id$. Define
$\mu : \pi_1|_{\mathscr{C}} \to \Delta_{\pi_1(X,x_0)}$ to be the natural transformation where components are $\mu_U = \pi_1(i : U \hookrightarrow X)$. A comparison of components gives the following commutative diagrams of functors from $\mathscr{C}$ to *Groupoid*:

$$\begin{tikzcd}
\pi_1|_{\mathscr{C}} \arrow[d, "\mu"'] & \Pi|_{\mathscr{C}} \arrow[l, "F"'] \arrow[d, "\gamma"] \\
\Delta_{\pi_1(X,x_0)} & \Delta_{\Pi(X)} \arrow[l, "F_X"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\Pi|_{\mathscr{C}} \arrow[d, "\gamma"'] & \pi_1|_{\mathscr{C}} \arrow[l, "J"'] \arrow[d, "\mu"] \\
\Delta_{\Pi(X)} & \Delta_{\pi_1(X,x_0)} \arrow[l, "J_X"]
\end{tikzcd}$$

combining the results above we have the following:

$$\begin{tikzcd}
\pi_1|_{\mathscr{C}} \arrow[dd, "\mu"'] \arrow[dr, "\xi"] & & \Pi|_{\mathscr{C}} \arrow[ll, "F"'] \arrow[dl, "\eta"'] \arrow[dd, "\gamma"] & & \pi_1|_{\mathscr{C}} \arrow[ll, "J"'] \arrow[dd, "\mu"] \arrow[llll, bend right, "Id"'] \\
& \Delta_G & & & \\
\Delta_{\pi_1(X,x_0)} \arrow[ur, dashed, "\tilde{\xi}"] & & \Delta_{\Pi(X)} \arrow[ll, "F_X"'] \arrow[ul, dashed, "\tilde{\eta}"'] & & \Delta_{\pi_1(X,x_0)} \arrow[ll, "J_X"'] \arrow[llll, bend left, "Id"]
\end{tikzcd}$$

Given group G and $\xi : \pi_1|_{\mathscr{C}} \to \Delta_G$, let $\eta = \xi \circ F$. By the groupoid version of the theorem, $\exists!\, \tilde{\eta} : \Delta_{\Pi(X)} \to \Delta_G \; s.t. \; \tilde{\eta} \circ \gamma = \eta$.
Uniqueness: if $\tilde{\xi} \circ \mu = \xi$ then $\tilde{\xi} \circ F_X \circ \gamma = \tilde{\xi} \circ \mu \circ F = \xi \circ F = \eta$, so $\tilde{\xi} \circ F_X = \tilde{\eta}$ and $\tilde{\xi} = \tilde{\eta} \circ J_X$.
Existence: $\tilde{\eta} \circ J_X \circ \mu = \tilde{\eta} \circ \gamma \circ J = \eta \circ J = \xi \circ F \circ J = \xi$;
Thus $\tilde{\xi} = \tilde{\eta} \circ J_X$ is the unique map of cocones required,
$\pi_1(X, x_0) = colim_{U\in C}\pi_1(U, x_0)$.

## The Fundamental Groupoid Version

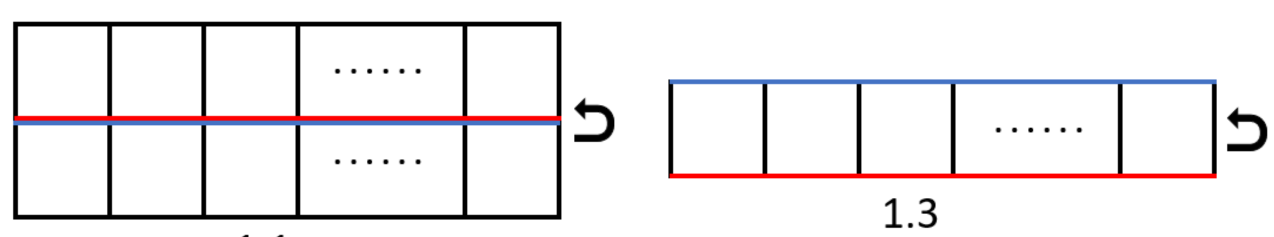
Before tackling $\pi_1(X, x_0)$, let us consider $\Pi_1(X)$. We can view $C$ as a subcategory $\mathscr{C}$ of Top, where objects are the elements of $C$ with the subspace topology, and morphisms are the inclusion maps. The restriction of the fundamental groupoid functor $\Pi_1$ to $\mathscr{C}$ gives a diagram of groupoids, which has the colimit $\Pi_1(X) \cong colim_{U\in\mathscr{C}}\Pi_1(U)$.
To see the result above, given a groupoid $GP$ and natural transformation $\eta : \Pi_1|_{\mathscr{C}} \to \Delta_{GP}$($\Delta_{GP}$ denotes the constant functor at $GP$), we want to find a unique $\tilde{\eta} : \Pi_1(X) \to GP$ such that $\tilde{\eta} \circ \gamma = \eta$, where $\gamma : \Pi_1|_{\mathscr{C}} \to \Delta_{\Pi_1(X)}$ is the natural transformation with components $\gamma_U = \Pi_1(i : U \hookrightarrow X)$.
• On objects, we must have $\tilde{\eta}(x) = \eta_U(x)$ for $x \in U \in C$.
• On morphisms, given a path $f$ in $X$, using the Lebesgue number lemma we can subdivide [0, 1] into n sub-intervals such that the image of each subinterval $I_k$ under $f$ lies in some $U_k \in C$. Reparametrizing each $f|_{I_k}$ to a path $f_k$, we decompose $[f] = [f_1 * f_2 * ... * f_n] = \circ_{k=1}^n [f_k]$, and

$$\tilde{\eta}([f]) = \circ_{k=1}^n \tilde{\eta}([f_k]) = \circ_{k=1}^n \eta_{U_k}([f_k]).$$

The above proof for uniqueness almost gives a definition for $\tilde{\eta}$. We just need to verify that it is well-defined.
• On objects, if $x \in U \in C$ and $x \in V \in C$, then $x \in U \cap V \in C$, and $\eta_U(x) = \eta_{U\cap V}(x) = \eta_V(x)$ by naturality of $\eta$.
• On morphisms, if a path $f$ lies entirely in both $U \in C$ and $V \in C$, then $f$ lies in $U \cap V \in C$, and $\eta_U([f]) = \eta_{U\cap V}([f]) = \eta_V([f])$ by naturality of $\eta$.
Generally, given paths $f$ and $g$ in $X$ with $[f] = [g]$, suppose we decompose them as $[f] = [f_1 * f_2 * ... * f_n]$ and $[g] = [g_1 * g_2 * ... * g_m]$, where each $f_i$ is a path in $U_i \in C$ and each $g_j$ is a path in $V_j \in C$. We want to prove that

$$\circ_{i=1}^n \eta_{U_i}([f_i]) = \circ_{j=1}^m \eta_{V_j}([g_j]).$$

Since $[f] = [g]$ we can find a path homotopy $H : I \times I \to X$ from $f$ to $g$. Using the Lebesgue number lemma again we can subdivide $I \times I$ into $(mnr)^2$ sub-squares of equal size such that the image of each $I_i \times I_j$ under H lies entirely in some $W_{i,j} \in C$.
For $i \in$ {1,2,...,mnr} and $j \in$ {0,1,...,mnr}, define $p_{i,j} : I \to X$ by $p_{i,j}(s) = H(mnrs - i + 1, \frac{j}{mnr})$. These paths correspond to horizontal edges of the sub-squares;
For $i \in$ {0,1,...,mnr} and $j \in$ {1,2,...,mnr}, define $q_{i,j} : I \to X$ by $q_{i,j}(s) = H(\frac{i}{mnr}, mnrs - j + 1)$. These paths correspond to vertical edges of the sub-squares. It follows that
$\circ_{i=1}^n \eta_{U_i}([f_i]) = \circ_{i=1}^{mnr} \eta_{W_{i,1}}([p_{i,0}])$ and $\circ_{j=1}^m \eta_{V_j}([g_j]) = \circ_{i=1}^{mnr} \eta_{W_{i,mnr}}([p_{i,mnr}])$.
Now it suffices to show that the right hand sides are equal.
We have the following relations:
(a) Observe that for any $i, j \in$ {1,2,...,mnr}, $p_{i,j}(I) \subseteq W_{i,j} \cap W_{i,j+1}$, so $\eta_{W_{i,j}}([p_{i,j}]) = \eta_{W_{i,j+1}}([p_{i,j}])$, hence $\circ_{i=1}^{mnr} \eta_{W_{i,j}}([p_{i,j}]) = \circ_{i=1}^{mnr} \eta_{W_{i,j+1}}([p_{i,j}])$ for $j \in$ {1,2,...,mnr}. (1.1)
(b) For any $j \in$ {0,1,...,mnr}, $q_{0,j+1} * p_{1,j}$ is homotopic to $p_{1,j+1} * q_{1,j+1}$, so
$\eta_{W_{1,j+1}}([q_{0,j+1}]) \circ \eta_{W_{1,j+1}}([p_{1,j}]) = \eta_{W_{1,j+1}}([p_{1,j+1}]) \circ \eta_{W_{1,j+1}}([q_{1,j+1}])$
$= \eta_{W_{1,j+1}}([p_{1,j+1}]) \circ \eta_{W_{2,j+1}}([q_{1,j+1}])$. (1.2)
After mnr similar operations we get $\circ_{i=1}^{mnr} \eta_{W_{i,j+1}}([p_{i,j}]) = \circ_{i=1}^{mnr} \eta_{W_{i,j+1}}([p_{i,j+1}])$. (1.3)
Repetitively apply results (a) and (b) to $\circ_{i=1}^{mnr} \eta_{W_{i,1}}([p_{i,0}])$, and we have the desired equality $\circ_{i=1}^{mnr} \eta_{W_{i,1}}([p_{i,0}]) = \circ_{i=1}^{mnr} \eta_{W_{i,mnr}}([p_{i,mnr}])$.

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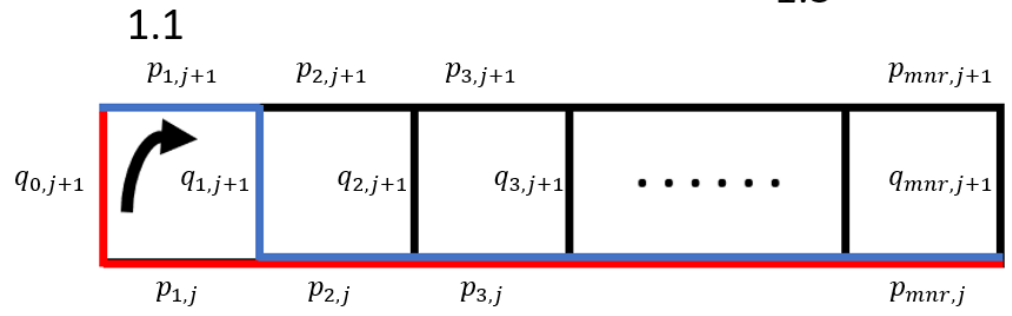


1.2

## References

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[2] Hatcher, A.(2009). *Algebraic Topology.* Cambridge University Press.