Geometric Triangulation Spaces Of Simple 3-Manifolds

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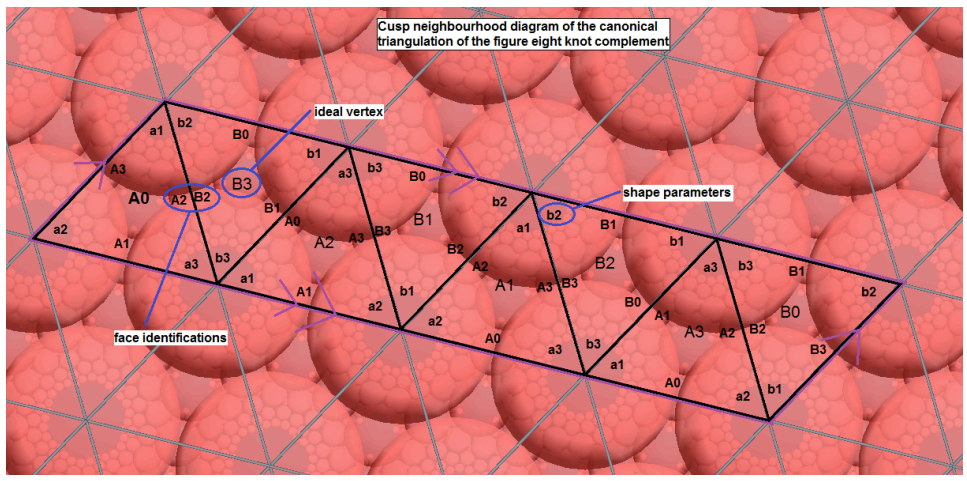
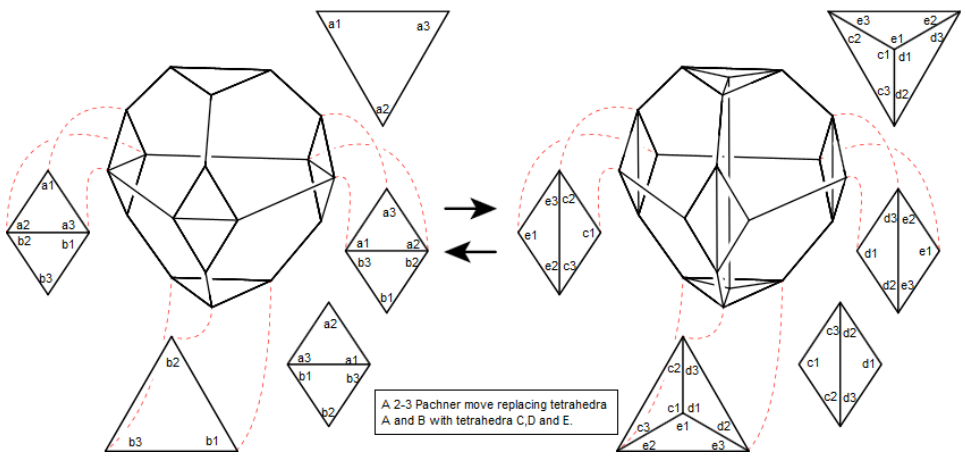
What is a geometric triangulation of a 3-manifold?

A 3-manifold is a topological space which looks locally like Euclidean 3-space. A triangulation of a 3-manifold is a combinatorial description of that manifold in terms of a set of tetrahedra with faces of tetrahedra identified in pairs. There are many distinct triangulations for a particular
3-manifold. If the triangulation consists of only positively oriented, ideal tetrahedra in hyperbolic space fitting together correctly, then it is possible to construct hyperbolic structure on the manifold. We then say the triangulation is geometric.



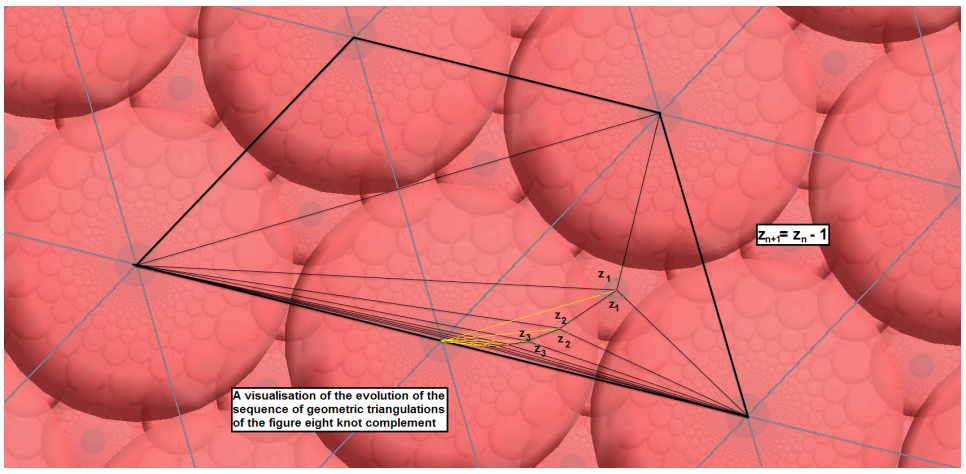
What was our project?

Given a particular triangulation of a 3-manifold, a distinct triangulation can be generated by taking two tetrahedra which have a common face and replacing them with three tetrahedra with a common edge. This is known as a 2-3 Pachner move (see below). While it was already known that any two distinct triangulations of the same manifold are connected by a number of 2-3 Pachner moves and their inverses, not much was known about the connectedness of the subset of geometric triangulations of 3-manifolds or the abundance or scarcity of geometric triangulations in the space of all triangulations of a 3-manifold. Our project was to investigate the spaces of geometric triangulations of some simple 3-manifolds, including the figure eight knot complement, its sister manifold and other once-punctured torus bundles, using both pencil and paper and computer assisted approaches to the problem.



What did we find?

For the figure eight knot complement we successfully found, and proved the existence of, a sequence of moves which gives an infinite path of geometric triangulations connected to its canonical triangulation. We were also able to completely describe all tetrahedral shapes for any triangulation in this sequence. While the figure eight knot sister manifold has a very similar structure to the figure eight knot complement manifold, we found that no such sequence existed for the sister. In fact, there were only two additional geometric triangulations connected to the sister manifold's canonical triangulation. Once-punctured torus bundles are a family of 3-manifolds related to the figure eight knot complement. We also investigated some members of this family. For the next simplest example in this family, known as m009, we used computational programs to identify the only possibly infinite sequence of geometric 2-3 Pachner moves. We then investigated this path closely on paper and again were able to prove the existence of infinitely many connected geometric triangulations for this manifold. Additional computations also suggest that for many other once-punctured torus bundles there are similar sequences of moves which result in infinitely many connected geometric triangulations. We have identified likely sites of convergence for these sequences to investigate further.



What did we gain from the experience?

The vacation scholar experience has been invaluable. We have gotten first hand insight into what mathematics research is like, met and worked with great people, developed our computer-assisted and pencil and paper problem solving approaches and learnt a lot of new mathematics.